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DEVELOPMENT OF METHODS FOR SOLVING TORSION PROBLEMS OF PHYSICALLY NONLINEAR SOLIDS

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The mechanical properties of many materials, such as concrete, cast iron, rocks, some structural graphites, refractory ceramics, etc., which are usually porous materials with an inhomogeneous structure, depend on the type of stress state. It is shown in the absence of unified diagrams of the relationship between stress and strain intensities for various types of stress state. Such dependence is typical for materials characterized by the growth of deformation in the nonlinear region of deformation. For these materials, the processes of volumetric and shear deformations are interrelated, which are expressed in the appearance of volumetric deformations during torsion. When the linear constitutive relations are used to analyze the torsion problems of such materials, a significant error occurs. The parameter characterizing the type of stress state can be, for example, the ratio of the average stress to the stress intensity. This paper considers the linear constitutive relations, which take into account the dependence of the mechanical properties of the material on the type of stress state. The results of numerical solution of a circular tube torsion problem by reducing it to a system of ordinary differential equations are presented. The system of differential equations is solved using the 4-order Runge–Kutta method with automatic step selection and error estimation. The method features implementations are discussed. In the second part of the article, the results of numerical modeling of circular tube torsion problems are described using a finite element analysis software, for which a special library that implements the considered constitutive relations is written. The features of the finite element analysis which were taken into account when writing the library code are shown. The calculation results demonstrate the presence of axial deformation during torsion. The results obtained by different methods are compared.

Key words*:* stress state type, torsion, Runge–Kutta method, FEM

1. Introduction

When studying some materials, it is found that their mechanical properties depend on the type of stress state, which is expressed in the dependence of the deformation diagrams on the ratio between the components of the stress tensor during loading [1-5]. For such materials the use of linear constitutive equations for modelling the mechanical behavior under certain types of loading, in particular during torsion, can lead to significant errors. Such materials include rocks, concrete. As a rule, mechanical properties of rocks can be described by Mohr–Coulomb model [6]. Experiments show that certain parameters of this model, such as the dilation angle and the modulus of elasticity, depend on external pressure [7]. There are various approaches that allow us to take these features into account, for example, by introducing additional dependencies [8]. In many studies, there are elaborated constitutive equations which take into consideration the parameters of the mechanical state, for example, damage [9-12]. Some of these models are applicable for the computer modeling of rock deformation processes [13-15], which is a question of high importance.

In the works [16-19], the constitutive equations of special type are proposed. These constitutive equations consider the dependence of the material mechanical properties on the type of stress state. Some authors [20, 21] also take into account the difference of elastic moduli in tension and compression. Such difference in modularity is usually understood as a change in the slope of the deformation diagram by a jump when the stress sign changes under uniaxial loading conditions. In reality, the deformation diagrams are smooth. Therefore, such piecewise linear approximations of weakly nonlinear diagrams should be treated as a mathematical approximation of real diagrams. Such approximation makes it possible to obtain analytical solutions to problems that can be used for verifying the results of numerical modeling. Thus, the term "multi-modularity" refers to the mathematical idealization of a smooth in reality deformation diagram under uniaxial loading conditions. At the same time, experiments on complex proportional loading (e.g., biaxial compression [22, 23], compression of a cylinder under the external pressure [2]) reveal the dependence of deformation diagrams on applied external pressure or on the stress state type.

Due to the complexity of the constitutive relations, it is not possible to obtain the analytical solutions to most of the practically important problems. The goal of this article is to consider various methods of numerical solution of torsion problems of physically nonlinear bodies with properties depending on the type of stress state. The used constitutive equations make it possible to apply several methods for numerical solution of

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problems. So, for a one-dimensional problem of a long tube torsion with a cylindrical side surface a partial numerical-analytical method of solution is possible. At first, the solution of the problem is carried out by analytical methods with simplifications and is reduced to solving a system of ordinary differential equations; then the resulting system is implemented by any convenient numerical method. In this case, the boundary conditions for the problem are satisfied only in integral form. With another method, the solution is found using the finite element method. Here, the downside is a large computational complexity at a step of the computational grid comparable to the numerical solution of a system of ordinary differential equations.

Some theoretical aspects of the applied numerical methods are analyzed in the paper: an algorithm for solving a system of differential equations by the Runge–Kutta method with automatic step selection and error estimation is described; a method is proposed for circumventing the problem of uncertainty of the Jacobian defining the relations at zero; a library for the FEM package is written, as well as a program for solving a system of ordinary differential equations. Verification of the obtained results was carried out.

2. Constitutive equations and torsion problem statement

The equations of the stress-strain relationship for an isotropic body experiencing torsion can be obtained on the basis of an appropriate representation of the potential function [16, 17]:

$$\Phi = (1/2)\left(1+\zeta(\xi)\right)\left(A+B\xi^2\right)\sigma_0^2, \tag{1}$$

where $\zeta(\xi) = \left(1+\kappa(\xi)\right)\left(A+B\xi^2\right)^{-1}$, $\xi = \sigma/\sigma_0$ is the stress state parameter, $\sigma = (1/3)\sigma_{ii}$ is the hydrostatic component of stresses, $\sigma_0 = \sqrt{(3/2)S_{ij}S_{ij}}$ is the equivalent stress, $S_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ is the stress deviator. The ξ parameter characterizes the stress state on average, because σ is the average normal stress at the point of the medium, and σ_0 is the average tangential stress at the same point [20]. When $\zeta(\xi) = 0$ equation (1) coincides with the potential function of a linear elastic body, where $A = \dfrac{2(1+\nu)}{3E}$, $B = \dfrac{3(1-2\nu)}{E}$. From (1), it is possible to get the stress-strain relationship equations:

$$\begin{gathered}
\varepsilon_{ij} = \frac{\partial\Phi}{\partial\sigma_{ij}} = (3/2)\omega(\xi)S_{ij} + (1/3)\Omega(\xi)\sigma\delta_{ij}, \\
\omega(\xi) = (-1/2)\left(A+B\xi^2\right)\zeta'(\xi)\xi + A\left(1+\zeta(\xi)\right), \\
\Omega(\xi) = (1/2)\left(A+B\xi^2\right)\zeta'(\xi)/\xi + B\left(1+\zeta(\xi)\right).
\end{gathered} \tag{2}$$

For the $\omega(\xi)$ and $\Omega(\xi)$ functions we can find the following relations:

$$\omega(\xi) + \xi^2\Omega(\xi) = \left(A+B\xi^2\right)\left(1+\zeta(\xi)\right), \quad \omega'(\xi) + \xi^2\Omega'(\xi) = 0. \tag{3}$$

Introducing the new parameter $\gamma = \varepsilon/\varepsilon_0$, it is possible to resolve the relations (2) with respect to the stresses: $\varepsilon = \varepsilon_{ii}$ is volumetric strain in case of small deformations, $\varepsilon_0 = \sqrt{(2/3)e_{ij}e_{ij}}$ is equivalent strain, $e_{ij} = \varepsilon_{ij} - (1/3)\varepsilon\delta_{ij}$ is strain deviator. It is also possible to find the relations between γ and ξ:

$$\gamma = \xi\left[\Omega(\xi)/\omega(\xi)\right]. \tag{4}$$

On the basis of (3) and (4), the equations (1) and (2) can be rewritten as follows:

$$U=(1/2)\left(1+\eta(\gamma)\right)\left(\frac{1}{A}+\frac{\gamma^2}{B}\right)\varepsilon_0^2,$$
$$\sigma_{ij}=(2/3)\psi(\gamma)e_{ij}+\Psi(\gamma)\varepsilon\delta_{ij},$$
$$\psi(\gamma)=(-1/2)\left(\frac{1}{A}+\frac{\gamma^2}{B}\right)\eta'(\gamma)\gamma+\frac{1}{A}\left(1+\eta(\gamma)\right), \tag{5}$$
$$\Psi(\gamma)=(1/2)\left(\frac{1}{A}+\frac{\gamma^2}{B}\right)\eta'(\gamma)\gamma^{-1}+\frac{1}{B}\left(1+\eta(\gamma)\right).$$

In case of the linear approximation of function $\omega(\xi)=A+C\xi$, constitutive equations (5) will take the form:

$$\sigma_{ij}=\left((2/3)(B-C\gamma)e_{ij}+(A-C/\gamma)\varepsilon\delta_{ij}\right)\left(AB-C^2\right)^{-1}, \tag{6}$$

where $\gamma=\varepsilon/\varepsilon_0$ is the strain state parameter, $e_{ij}=\varepsilon_{ij}-(1/3)\varepsilon\delta_{ij}$ is the strain deviator, $\varepsilon=\varepsilon_{ii}$ is the volumetric strain, C is a parameter characterizing the degree of sensitivity of the material properties to the type of stress state of the material. When $C=0$, the constitutive relations (6) are reduced to the Hooke's law.

This paper provides a solution to the problem of unconstrained torsion of a thick-walled tube made of a material with properties depending on the type of stress state. The conditions at the boundaries are considered: a torque acts in the end sections of the tube; there is no axial force; the lateral cylindrical surface is stress-free.

After transfer to non-dimensional variables, the parameters of the problem become as follows: internal $a/a=1$ and external $r_*=b/a=2$ radii; stress values multiplied by A; the angle of twisting of the end section α is related to the radius a .

3. Solving the torsion problem by reducing to a system of ordinary differential equations

The classical solution of torsion problems of cylindrical bodies, based on the Saint-Venant's principle, is not suitable for materials sensitive to the type of stress state due to the relationship of volumetric and shear deformations [18]. Therefore, in order to construct solutions to torsion problems, it is necessary to consider more general representations for displacements in cylindrical bodies for the cases when deformations do not depend on the longitudinal coordinate. Such representations are formulated in [17] based on the analysis of the Saint-Venant's compatibility conditions and are valid for various materials with any mechanical properties.

Taking into account the axial symmetry of the problem, displacement in cylindrical coordinates can be represented as functions of the radius and the longitudinal coordinate regardless of the angle θ [17]:

$$u_r=\phi_1(r,z),\quad u_\theta=\phi_2(r,z),\quad u_z=\phi_3(r,z). \tag{7}$$

Due to the invariability of the boundary conditions, as well as the fact that the tube is considered long enough, we have:

$$\frac{\partial\varepsilon_{ij}}{\partial z}=0,\qquad \frac{\partial\sigma_{ij}}{\partial z}=0. \tag{8}$$

The components of the strain tensor are associated with the formulas introduced in equations (7):

$$\varepsilon_{rr}=\frac{\partial\phi_1}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{\phi_1}{r},\quad \varepsilon_{zz}=\frac{\partial\phi_3}{\partial z},\quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{\partial\phi_2}{\partial r}-\frac{\phi_2}{r}\right),\quad \varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial\phi_1}{\partial z}+\frac{\partial\phi_3}{\partial r}\right),\quad \varepsilon_{\theta z}=\frac{1}{2}\frac{\partial\phi_2}{\partial z}.$$

We determine the nature of the dependence of the functions ϕ_1, ϕ_2, ϕ_3 on the longitudinal coordinate z .

From the equations of the Saint-Venant's compatibility condition follows:

– $\partial\varepsilon_{zz}/\partial r=0$;

– from the relation of the condition $\partial\varepsilon_{zz}/\partial z = 0$ follows that $\varepsilon_{zz} = \beta$, $\phi_3(r,z) = f_3(r) + \beta z$;
– from the relation of the condition $\partial\varepsilon_{rz}/\partial z = 0$ and taking into account the expression for ϕ_3 follows that $\dfrac{\partial^2\phi_1}{\partial z^2} = 0$, $\phi_1 = k_1(r)z + f_1(r)$, $\dfrac{\partial\varepsilon_{\theta\theta}}{\partial z} = \dfrac{1}{r}\dfrac{\partial\phi_1}{\partial z} = 0 \Rightarrow k_1(r) = 0$.

Then a function ϕ_2 can be found out, since $\dfrac{\partial\varepsilon_{\theta z}}{\partial z} = \dfrac{1}{2}\dfrac{\partial^2\phi_2}{\partial z^2} = 0.$ Thus, $\phi_2 = k_2(r)z + f_2(r)$. At the same time,

$$\frac{\partial\varepsilon_{r\theta}}{\partial z} = \frac{1}{2}\left(\frac{\partial^2\phi_2}{\partial r\partial z} - \frac{1}{r}\frac{\partial\phi_2}{\partial z}\right) = 0,$$

$$k_2'(r) - \frac{k_2'(r)}{r} = 0,$$

$$k_2 = \alpha r.$$

As a result, expressions for displacements (7) are converted to the form:

$$u_r = f_1(r), \quad u_\theta = \alpha rz + f_2(r), \quad u_z = \beta z + f_3(r).$$

In these expressions, opposite to (7), the dependencies on the longitudinal coordinate z are presented explicitly. Let us find the relationships between the functions $f_1(r)$, $f_2(r)$, $f_3(r)$. We express the components of the strain tensor through them:

$$\varepsilon_{rr} = f_1'(r), \quad \varepsilon_{\theta\theta} = \frac{f_1(r)}{r}, \quad \varepsilon_{zz} = \beta,$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(f_2'(r) - \frac{f_2(r)}{r}\right), \quad \varepsilon_{rz} = \frac{1}{2}f_3'(r), \quad \varepsilon_{\theta z} = \frac{1}{2}\alpha r.$$

Since now the deformations depend only on r , the stresses, according to the constitutive relations, also depend only on r . In this case, the equilibrium equations take the form:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{d\sigma_{r\theta}}{dr} + \frac{2\sigma_{r\theta}}{r} = 0, \quad \frac{d\sigma_{rz}}{dr} + \frac{\sigma_{rz}}{r} = 0. \tag{9}$$

After integrating the second and third equations from (9), we obtain:

$$\sigma_{r\theta} = C_1/r^2, \quad \sigma_{rz} = C_2/r .$$

In general, the boundary conditions $T_i = \sigma_{ij}n_j$ on a cylindrical surface take the form:

$$T_1 = \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta = 0, \quad T_2 = \sigma_{rr}\sin\theta + \sigma_{r\theta}\cos\theta = 0, \quad T_3 = \sigma_{rz} = 0 .$$

It follows from these conditions that at $r = b$ stresses $\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0$. As a result, $C_1 = C_2 = 0$, and, therefore, $\sigma_{r\theta} = \sigma_{rz} = 0$ in the entire pipe. From the constitutive equation (6) we can find the expression for strain tensor components: $\varepsilon_{r\theta} = \varepsilon_{rz} = 0$ and get the equations for determining the functions $f_2(r)$, $f_3(r)$:

$$f_2' - f_2 r^{-1} = 0, \quad f_3' = 0 . \tag{10}$$

According to equations (10), the functions are: $f_2 = C_3 r$, $f_3 = C_4$. If at least one point of the tube cross-section at $z = 0$ is fixed, then $C_3 = C_4 = 0$. Thus, the components of the strain tensor are expressed only in terms of one function $f_1(r)$, which we denote $f(r)$. In this case, the following formulas become:

– for the components of the strain tensor [17]:

$$\varepsilon_{rr} = f'(r), \quad \varepsilon_{\theta\theta} = \frac{f(r)}{r}, \quad \varepsilon_{zz} = \beta, \quad \varepsilon_{r\theta} = \varepsilon_{rz} = 0, \quad \varepsilon_{\theta z} = \frac{1}{2}\alpha r \,,$$
$$\varepsilon = f'(r) + f(r)/r + \beta, \quad \gamma = \varepsilon/\varepsilon_0 \,,$$
$$\varepsilon_0 = (2/3)\left[\varepsilon_r r^2 + \varepsilon_{\theta\theta}^2 + \varepsilon_{zz}^2 + 3\left(\varepsilon_{r\theta}^2 + \varepsilon_{\theta z}^2 + \varepsilon_{rz}^2\right) - \left(\varepsilon_{rr}\varepsilon_{\theta\theta} + \varepsilon_{\theta\theta}\varepsilon_{zz} + \varepsilon_{rr}\varepsilon_{zz}\right)\right]^{1/2} =$$
$$= (2/3)\left[\left(f'(r)\right)^2 + \frac{f^2(r)}{r^2} + \beta^2 + \frac{3\alpha^2 r^2}{4} - \frac{f'(r)\,f(r)}{r} - \frac{f(r)\beta}{r} - \beta f'(r)\right]^{1/2};$$

– for the stress tensor components (see constitutive relations (6))

$$\sigma_{rr} = \left((2/3)(B - C\gamma)\left(f'(r) - (1/3)\varepsilon\right) + (A - C/\gamma)\varepsilon\right)\Big/\left(AB - C^2\right),$$
$$\sigma_{\theta\theta} = \left((2/3)(B - C\gamma)\left(f(r)/r - (1/3)\varepsilon\right) + (A - C/\gamma)\varepsilon\right)\Big/\left(AB - C^2\right),$$
$$\sigma_{zz} = \left((2/3)(B - C\gamma)\left(\beta - (1/3)\varepsilon\right) + (A - C/\gamma)\varepsilon\right)\Big/\left(AB - C^2\right).$$

After substituting the obtained expressions into the equilibrium equations, we come to a nonlinear differential equation to determine the function $f(r)$:

$$\left(A + \frac{4}{9}B - \frac{4}{9}C\gamma\right) f'' - \frac{4}{9}Cf'\gamma' - \frac{\left(A - \frac{2}{9}B\right)(f'r - f)}{r^2} + \frac{2}{9}C\left[\frac{f}{r}\gamma' + \frac{(rf' - f)\gamma}{r}\right] + \frac{2}{9}C\beta\gamma' -$$
$$- \frac{C}{\gamma}\left(f'' + \frac{f'}{r} - \frac{f}{r^2}\right) + \frac{C\gamma'}{\gamma^2}\left(f' + \frac{f}{r} + \beta\right) + \frac{2}{3r}(B - C\gamma)\left(f' - \frac{f}{r}\right) = 0, \qquad (11)$$

where the function $\gamma'(r)$ is represented as:

$$\gamma' = \frac{1}{\varepsilon_0}\left(f'' + \frac{f'r - f}{r^2}\right) - \frac{2}{9\varepsilon_0^3}\left(f' + \frac{f}{r} + \beta\right)\left[2f''f' + \frac{2ff'}{r^2} - \frac{2f^2}{r^3} - \frac{ff''}{r} - \frac{rf' - f}{r^2}(f' + \beta) + \frac{3\alpha^2 r}{2} - \beta f''\right]. \qquad (12)$$

The differential equation (11), taking into account (12) and formulas for the components of the strain tensor, is solvable with respect to the highest derivative, so it can be represented as a system of two first-order differential equations. Let us introduce a substitution $y_1 = f(r)$, then the system is transformed to the form:

$$y_2 = y_1', \quad y_2' = -\frac{SS_1 + S_5 + S_6}{S_7 + S_8 + S_9},$$

where

$$S_1 = -\frac{C}{9}\left(4y_2 - \frac{2y_1}{r} - \frac{9k}{\gamma} - 2\beta\right),$$
$$S_2 = \frac{y_2 r - y_1}{kr^2}, \quad S_3 = \frac{2\gamma}{9k^2}, \quad S = S_2 - S_3 S_4,$$

$$S_4 = \frac{(2y_1 - y_2 r - \beta r)(y_2 r - y_1)}{r^3} + \frac{3\alpha^2 r}{2},$$

$$k = \frac{2}{3}\left(y_2^2 + \frac{y_1^2}{r^2} + \beta^2 + \frac{3\alpha^2 r^2}{4} - \frac{y_2 y_1}{r} - \frac{y_1 \beta}{r} - \beta y_2 \right)^{1/2},$$

$$S_5 = \frac{\left(-\frac{C}{\gamma} + A - \frac{2}{9}B + \frac{2C\gamma}{9} \right)(y_2 r - y_1)}{r},$$

$$S_6 = \frac{2(B - C\gamma)(y_2 r - y_1)}{2r^2}, \qquad \lambda = \frac{4C\left(k + \frac{\gamma\beta}{9} \right)}{9k^2 r},$$

$$S_7 = A + \frac{4}{9}B - \frac{4C\gamma}{9} - (2y_2 r - y_1)\lambda,$$

$$S_8 = 2C\left[\frac{\beta}{9k} + \frac{2(2y_2 r - y_1)^2 \gamma}{(9kr)^2} \right],$$

$$S_9 = 2C\left[\beta r\left(\frac{1}{k} + \frac{2\beta\gamma}{9} \right) - \frac{\gamma(2y_2 r - y_1)}{9k^2 r} \right].$$

The system of equations will not change when moving to dimensionless variables: $\bar{r} = r/a$, $\bar{y}_1 = y_1/a$, $\bar{\alpha} = \alpha a$ and dimensionless stresses $\bar{\sigma}_{ij} = \sigma_{ij} A$. In the following text, we omit the dash in the notation of variables.

The boundary conditions for the tube torsion problem have the form:

$$\sigma_{rr}\big|_{r=1} = 0, \qquad \sigma_{rr}\big|_{r=r*} = 0.$$

The values of the torque and axial force are found during the solution according to the formulae:

$$M = 2\pi \int_1^{r*} \sigma_{\theta z} r^2 dr, \quad F_z = 2\pi \int_1^{r*} \sigma_{zz} r dr .$$

The parameter β, characterizing the axial deformation can be calculated from the condition that the axial force is equal to zero [17].

To numerically solve the problem in a differential formulation and determine two necessary conditions at one point, the shooting method and the Runge–Kutta method with automatic selection of the integration step and error estimation were used [21-23]. To solve the problem, the following parameter values are selected: $B/A = 5/3$, $C/A = 0.6$, $\alpha = 0.015$, $r_* = b/a = 2$.

Let us use the equations of the Runge–Kutta method of the 4th order with the coefficients given in [23]. Consider the Cauchy problem:

$$y' = f(x, y), \qquad y(x_0) = y_0. \tag{13}$$

The approximate value at the subsequent points in the step h is calculated by the formula:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where:

$$k_1 = f(x_n, y_n),$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right),$$
$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right),$$
$$k_4 = f\left(x_n + h, y_n + hk_3\right).$$

To evaluate the calculation error and the efficiency of the integration step selection, we will use the horizontal step selection method. Let us calculate the values of the integral at a point in one and two steps, taking into account the error of the method, then find their difference:

$$\Delta = \left(I_1 - I_{22}\right)\big/\left(1 - 1/2^s\right),$$

where I_1 is the value of the integral calculated in one step, I_{22} is the value of the integral in two steps, s is the order of the method (in the case under consideration $s=4$). Then the main error term will be equal to: $\Delta = Ch^{s+1}$, where C is a constant.

We choose a new step h_{new} so that the error on it is $Ch_{new}^{s+1} = \epsilon$:

$$\left(h/h_{new}\right)^{s+1} = \Delta/\epsilon = \chi.$$

The next step is defined as

$$h_{new} = 0.95h\big/\chi^{1/(s+1)}.$$

The general error is calculated by the formula:

$$\delta_{k+1} = \delta_k \exp\left(\int_{x_k}^{x_{k+1}} \mu dx\right) + r_k,$$

where $\left(x_{k+1} - x_k\right)$ is the size of the $k+1$-th step, δ_k is the value of global error to the k-th step, r_k is the local error at the k-th step, μ is the maximum singular number which is equal to the maximum eigenvalue of the matrix $\left(J + J^{\mathrm{T}}\right)\big/2$, where J is the Jacobian of the system (13).

As a result of computational experiments, it was found that the maximum general error of the solution does not exceed $1\cdot 10^{-6}$. The distributions of strain and stress tensor components over the tube thickness are shown in Figure 1. Figure 1a demonstrates that the values of volume and shear deformations are comparable.

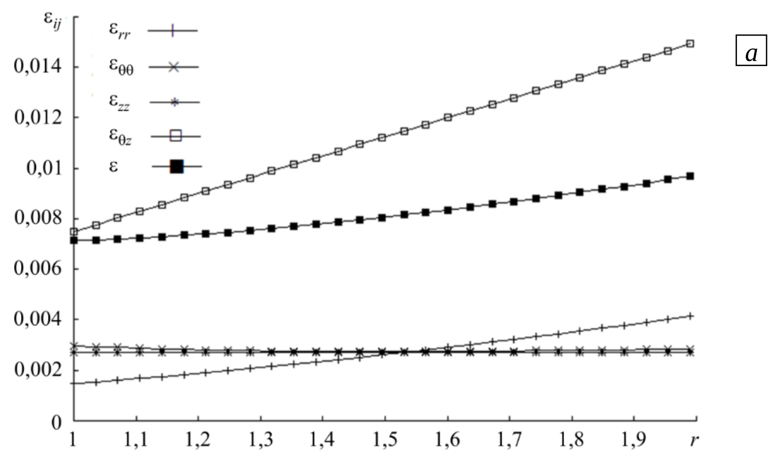

Fig. 1. Distributions of strain tensor components (a) and stress (b) in thickness for the pipe torsion problem when calculated by the method of reduction to a system of ordinary differential equations; dimensionless stresses considered (multiplied by a parameter A).

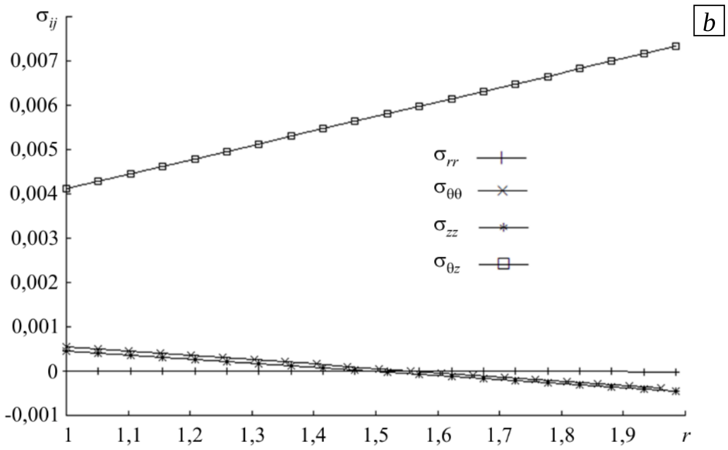

Fig. 1. *Continued*

In addition, the calculation results show that torsion is accompanied by axial deformation. Also, from the analysis of stress curves (Fig. 1b) it can be concluded that taking into account the physical nonlinearity during torsion leads to a significant difference from the case of linear elasticity: in the classical solution, only one shear component of the stress tensor is non-zero.

4. Solving the torsion problem using the finite element method

A finite element analysis package was used to obtain a numerical solution to the torsion problem. In order to implement the defining relationships into a package, it is required to write a custom implementing function based on USERMAT [28, 29] and the corresponding connection code in a scripting language. As part of this work, a special library in FORTRAN 77 was written for these purposes.

So, according to the finite element method, the solution of the problem is reduced to solving a system of algebraic equations:

$$[K]\{u\} = F^a ,$$

$[K]$ is a matrix of coefficients, $\{u\}$ is a vector of unknown degrees of freedom, which is understood as a displacement vector, F^a is a vector of applied stresses. In the case when algebraic relations are nonlinear, direct methods of solving the system, for example, the Gauss method, are unacceptable. The FEM analysis system uses the Newton–Raphson iterative method:

$$\left[K_i^{\mathrm{T}}\right]\{\Delta u_i\} = \{F^a\} - \{F_i^{inr}\},$$
$$\{u_{i+1}\} = \{u_i\} + \{\Delta u_i\},$$

where $\left[K_i^{\mathrm{T}}\right]$ is Jacobian, i is the index indicating the current cycle iteration, $\{F_i^{inr}\}$ is the vector of internal stresses caused by internal deformations to the i-th cycle iteration.

The essence of the method is reduced to the sequential execution of the following steps:

1. Choose the $\{u_i\}$. Usually, as $\{u_i\}$ we take the converged solution from the previous step. On the first step we choose $\{u_0\} = \{0\}$.

2. Calculate the updated Jacobian and the internal stress vector according to the given $\{u_i\}$.

3. Find the difference relative to the value on the first step $\{\Delta u_i\}$.

4. Add $\{\Delta u_i\}$ to the $\{u_i\}$, to get the following approximation $\{u_{i+1}\}$.

5. Repeat the steps until the convergence conditions of the solution are met.

In fact, the solution is approximated by tangent segments. Consequently, the method will not converge in the case of large second derivatives $\{F_i^{inr}\}$. In addition, problems will arise in the case when the matrix $\left[K_i^{\mathrm{T}}\right]$

is degenerate, for example, in the sense of the conditionality number, which in general is found as the product of the norms of the original and inverse matrices: $\kappa(A) = \|A^{-1}\| \cdot \|A\|$,

The FEM analysis system provides the following conditions for the convergence of the method [17]:

$$\|\{R\}\| < \varepsilon_R R_{ref},$$
$$\|\{\Delta u_i\}\| < \varepsilon_u u_{ref},$$

Where the vector $\{R\} = \{F^a\} - \{F^{inr}\}$ is the stress residual vector, $\{\Delta u_i\}$ is the increment of displacements at the i-th step, ε_R and ε_u are the constants responsible for the accuracy of the method. Here a standard L_2 norm is used: $\|\{R\}\| = \left(\sum R_i^2\right)^{1/2}$. Usually, $R_{ref} = \|\{F^a\}\|$, $u_{ref} = \|\{u\}\|$.

At each iteration of the Newton–Raphson method, the USERMAT function is called. At the input, the values of the stress and strain tensors and state parameters at the beginning of the iteration in time, as well as the strain tensor increments, are transmitted to it from ANSYS. The USERMAT function corresponding to the torsion problem being solved should update the stress tensors and state variables to the state relevant at the end of the iteration. In addition, it must calculate the matrix of partial derivatives $\partial\sigma_{ij}/\partial\varepsilon_{kl}$.

It should be noted that stress, strain tensors and the matrix of partial stress derivatives $\partial\sigma_{ij}/\partial\varepsilon_{kl}$ are stored in vector or matrix forms. For the three - dimensional case the following order of components is accepted: 11, 22, 33, 12, 23, 13. For mixed components of the strain tensor we have: $\gamma_{ij} = 2\varepsilon_{ij}$, $i \neq j$.

At the zero step, the values $\varepsilon_{ij}^0 = 0$, $\Delta\varepsilon_{ij}^0 = 0$ pass to the input of the written USERMAT function. The peculiarity of the considered constitutive equations is that the values of the derivatives $\partial\sigma_{ij}/\partial\varepsilon_{kl}$ at zero deformations are not known, and, accordingly, it is not possible to calculate the matrix of partial derivatives at the zero step by definition. This difficulty can be circumvented if, at the zero step, the matrix of partial derivatives is considered to be the corresponding matrix for Hooke's law ($C = 0$ with the corresponding Lame parameters: $\lambda = \dfrac{1}{B} - \dfrac{2}{9A}$, $\mu = \dfrac{1}{3A}$).

The library is based on a standard example of the implementation of the USERMAT function (Hooke's law). Verification was carried out: at the value of the parameter $C = 0$ the resulting solution corresponds to a solution based on a linear elasticity model.

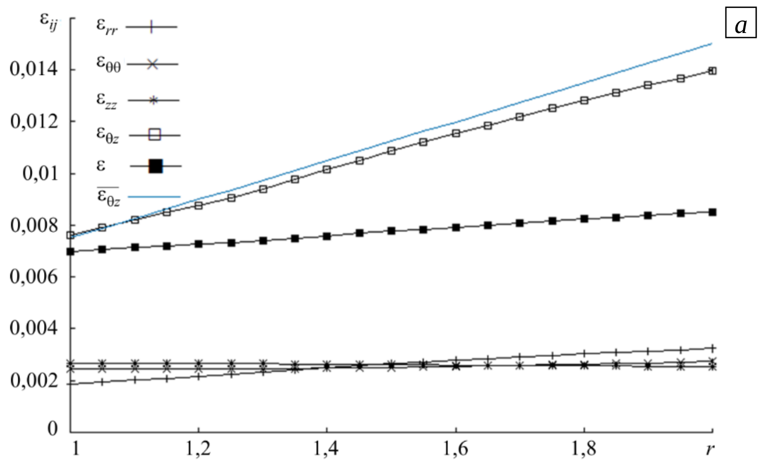

Fig. 2. Distributions of strain tensor components (a) and stresses (b) for the pipe torsion problem when calculated by the finite element method; dimensionless stresses multiplied by the parameter A considered; $\bar{\varepsilon}_{\theta z}$ and $\bar{\sigma}_{\theta z}$ – are the mixed components of the stress tensor for the case of linear elasticity.

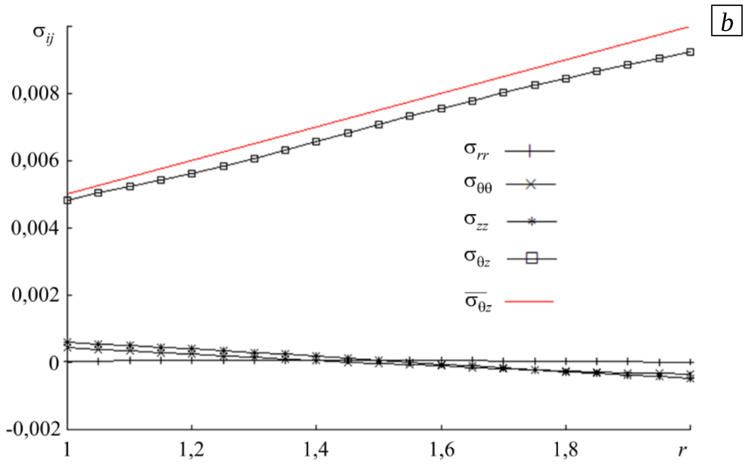

Fig. 2. *Continued.*

Let us perform the calculation for the torsion problem of a cylinder with an internal radius of 1, an external radius of 2, a length of 20 at $\frac{B}{A}=\frac{5}{3}$, $\frac{C}{A}=0{,}6$ and a twist angle $\alpha=0.015$ of one of the ends, the other end is rigidly fixed. The grid model of the computational domain was created by the command "SMRTSIZE, 1". For discretization, SOLID186 finite elements were used. These are volumetric (three-dimensional) hexahedral quadratic elements having twenty nodes with a quadratic representation of displacements. The results of the numerical solution are shown in Figure 2. For comparison, the distributions of the mixed components of the strain $\bar{\varepsilon}_{\theta z}$ and stress tensor $\bar{\sigma}_{\theta z}$ are also given for the case of linear elasticity ($C=0$), when $\varepsilon_{rr}=\varepsilon_{\theta\theta}=\varepsilon_{zz}=0$ and $\sigma_{rr}=\sigma_{\theta\theta}=\sigma_{zz}=0$.

After comparing Figure 1a with 2a and Figure 1b with 2b, it can be concluded that the results of solving the problem of unconstrained tube torsion obtained by different methods practically coincide qualitatively and quantitatively. Some differences can be explained by the fact that when solving using the finite element method, the grid was quite rough. It should also be noted that there is a non-zero axial deformation ε_{zz}.

5. Conclusions

On the basis of the numerical solution of the problems and analysis of the results, it is found that the use of the proposed constitutive equations makes it possible, while remaining in the conditions of the theory of small deformations, to describe the effect of volume increase during cylindrical and tubular samples torsion, as well as changes in the length and dimensions of the cross section. In contrast to the solution for a linearly elastic body, in which only two shear components of the stress tensor and two shear components of the strain tensor are non-zero, other components of the strain and stress tensors are non-zero for the materials considered.

Two methods for solving the tube torsion problem are tested:

1) by reducing to a system of ordinary differential equations and its subsequent numerical solution by the Newton–Raphson method;

2) using the finite element method. For the proposed defining relations, the material function library and the connection code to the finite element analysis system are written. Verification of the work of the written library for solving torsion problems was carried out. The results of computational experiments confirm that the considered constitutive relations are able to describe the behavior of a wide class of materials: rocks, concrete, structural graphite and others, the properties of which may change during loading.

The developed software can later be used to solve various applied tasks.

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