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ON THE STUDY OF VIBRATIONS OF A CYLINDER WITH A VISCOELASTIC COATING

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The paper presents the problem of steady-state longitudinal-radial vibrations of an elastic hollow cylinder with a viscoelastic coating. The properties of the coating change only along the radial coordinate and are described by the variable Lamé parameters and density. The ends of the cylinder are under sliding interface conditions, and the outer side surface is subject to periodic loads. Within the framework of the standard viscoelastic body model and following the principle of correspondence, the variable Lamé parameters are replaced by complex functions of the radial coordinate and vibration frequency. The solution was obtained using two approaches. In the first approach, the solution is found using the separation of variables method of and is reduced to a set of boundary value problems for canonical systems of first-order differential equations with variable coefficients. Further, each of these problems is obtained numerically by the shooting method. The second approach is based on the finite element method implemented in the FlexPDE package. We compared the obtained solutions for the given laws of variation of the Lamé parameters and a fixed frequency by analyzing graphs of the real and imaginary parts of radial displacement and stress components. The convergence of the solution found by the finite element method is shown as a function of the number of nodes for the values of the radial displacement functions measured at three points. Graphs of the amplitude-frequency characteristics measured on the outer surface are plotted for various values of the relaxation time. The effect of the variable properties of the coating on the displacement function is estimated. The advantages of each of the approaches and areas of their practical application are described.

Key words: cylinder, viscoelastic coating, functionally graded material, variable properties, vibrations, shooting method, finite element method

1. Introduction

Modern technologies make it possible to create the so-called functionally graded materials (FGM), the properties of which change along spatial coordinates [1–3]. They are obtained by mechanically joining materials with sharply different values of physicochemical parameters, and their main advantage, compared, for example, with conventional layered composites, is a significant reduction in the likelihood of cracks and delaminations. In addition, coatings made of them are able to change some characteristics of the object bearing them: corrosion resistance, strength, dielectric resistance, heat resistance, acoustic permeability, and others. Taking into account the variability of the elastic properties of the coating makes it possible to carry out more accurate calculations of the stress-strain state (SSS) for the «coating–object» system and to assess the strength. An important applied problem is also the analysis of the influence of the laws of material inhomogeneity on the acoustic properties of piecewise-homogeneous bodies that arise when operating in the mode of steady oscillations.

Cylindrical shapes are the most common elements of structures and utilities. An extensive review of works for 2005–2015 devoted to the mathematical description of the FGM structure is presented in the article [4]. One of the approaches to modeling the properties of such materials uses the classical theory of elasticity. In this case, elastic moduli are assumed to be variables in surface coordinates.

When solving problems for inhomogeneous objects, some assumptions are usually made. Among them are the following: Poisson's ratio is considered constant; all characteristics change according to one law; a particular type of inhomogeneity with a finite number of parameters is chosen. These starting points allow one in some cases to construct analytical solutions. In the case of a general type of inhomogeneity, the solution of the problem can only be found numerically. At the same time, at present, among the most common methods for obtaining approximate solutions, the finite element method (FEM) can be noted. At the same time, classical methods and approaches continue to be used for research in certain modern problems. For example, the problem of steady forced vibrations of an elastic homogeneous cylinder of finite length is solved using the

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method of separation of variables [5]. A similar method is used in the study of some symmetrical objects made from FGM [6].

In connection with the development of technology, increasing requirements for durability and efficiency of engineering structures, metal pipes coated with polymeric materials or entirely made from them (for example, from polyvinyl chloride or polyethylene) are becoming more common. The main advantage of these materials is corrosion resistance, low weight and ease of installation by soldering. Taking into account the variable properties of the coating material it allows for a more accurate calculation of the SSS of the product. The analysis of changes in the acoustic characteristics of a structure depending on the type of its inhomogeneity can be used in diagnostics during operation.

In this paper, we consider the problem of steady-state longitudinal-radial oscillations of a cylinder with an inhomogeneous viscoelastic coating. Two numerical approaches to the construction of solutions are implemented, which were previously tested in the study of an elastic cylinder without a coating [6]. The features and capabilities of each of the approaches are revealed.

2. Formulation of the problem

Let there be a hollow elastic cylinder with height $2h$ and known inner (r_1) and outer (r_2) radiuses. To solve the problem, we choose a cylindrical coordinate system $Or\varphi z$.

An inhomogeneous viscoelastic coating is deposited on the outer side surface of the cylinder in the region $D=\{(r,\varphi,z)\in R^3 \mid r\in[r_*,r_2],\ \varphi\in[0,2\pi],\ z\in[-h,h]\}$. The properties of the coating change only along the radial coordinate. The ends of the cylinder are in a sliding seal. There are no loads on the inner surface of the cylinder, and a normal distributed load is applied to the outer surface of the coating, changing along the coordinate z and causing longitudinal-radial oscillations of the system with a frequency ω (see Fig. 1).

Taking into account the axial symmetry of the geometry of the object under study, the type of loading and the law of change in the properties of the coating for the «cylinder–coating» system, we can write the equations of steady vibrations and boundary conditions in the form:

$$\begin{cases} \dfrac{\partial\sigma_{rr}}{\partial r}+\dfrac{\partial\sigma_{zr}}{\partial z}+\dfrac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r}+\rho\omega^2u_r=0, \\ \dfrac{\partial\sigma_{rz}}{\partial r}+\dfrac{\partial\sigma_{zz}}{\partial z}+\dfrac{\sigma_{rz}}{r}+\rho\omega^2u_z=0, \\ u_z=0, \quad \sigma_{rz}=0, \quad r\in[r_1,r_2], \quad z=\pm h, \\ \sigma_{rr}=\sigma_{rz}=0, \quad r=r_1, \quad z\in[-h,h], \\ \sigma_{rr}=-p(z), \quad \sigma_{rz}=0, \quad r=r_2, \quad z\in[-h,h], \end{cases}$$

where $\sigma_{rr}=\lambda\theta+2\mu\varepsilon_{rr}$, $\sigma_{zr}=\sigma_{rz}=2\mu\varepsilon_{zr}$, $\sigma_{\varphi\varphi}=\lambda\theta+2\mu\varepsilon_{\varphi\varphi}$, $\sigma_{zz}=\lambda\theta+2\mu\varepsilon_{zz}$ — components of the Cauchy stress tensor, $\varepsilon_{rr}=\dfrac{\partial u_r}{\partial r}$, $\varepsilon_{\varphi\varphi}=\dfrac{u_r}{r}$, $\varepsilon_{zz}=\dfrac{\partial u_z}{\partial z}$, $\varepsilon_{rz}=\dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial r}+\dfrac{\partial u_r}{\partial z}\right)$ — small strain tensor components, $\theta=\varepsilon_{rr}+\varepsilon_{\varphi\varphi}+\varepsilon_{zz}$ — volumetric deformation, λ, μ, ρ — variable material characteristics and density respectively, $u_r=u_r(r,z)$, $u_z=u_z(r,z)$ — displacement vector components in the radial and longitudinal directions, $p(z)=p^0q(z)$, p^0 и $q(z)$ — load amplitude and the law of its change. For definiteness, we assume that the function $q(z)$ is even and can be expanded into a series of the form

$$q(z)=\sum_{k=0}^{\infty}P_k\cos(\nu_k z),$$

in which the coefficients are calculated by the formulas: $P_0=\dfrac{1}{2h}\int_{-h}^{h}q(z)dz$, $P_k=\dfrac{1}{h}\int_{-h}^{h}q(z)\cos(\nu_k z)dz$, $\nu_k=\dfrac{\pi k}{h}$ ($k=1,2\ldots$).

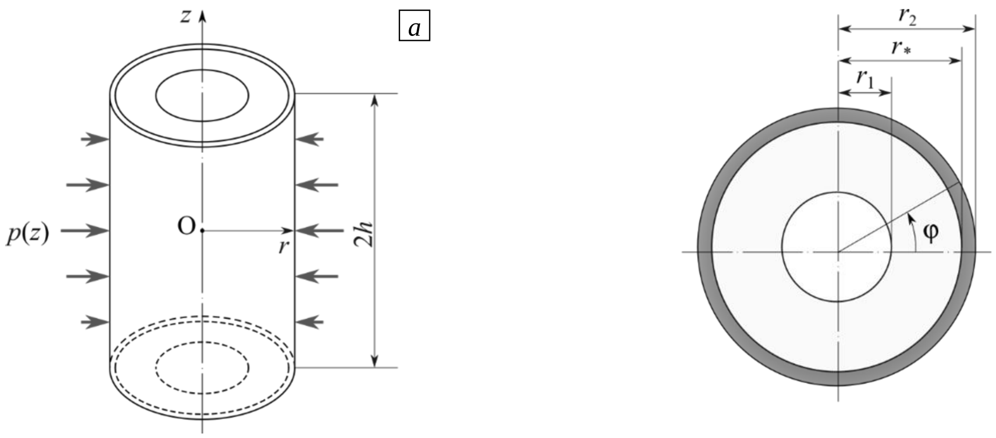


Fig. 1. Scheme for the formulation of the problem: a hollow cylinder with an inhomogeneous coating under the action of a normal distributed periodic load (*a*); cross section of the «cylinder–coating» system (*b*).

To describe the change in the viscoelastic properties of the coating along the radial coordinate, we use the three-parameter model of a standard viscoelastic body. Following the principle of correspondence, we represent the Lame parameters for the coating material as dimensionless complex functions of the radial coordinate and vibration frequency [1]:

$$\frac{\lambda(r,\omega)}{\lambda^*} = \Lambda(\xi,\kappa) = \frac{ib\kappa\varphi_{12}(\xi)+\varphi_{11}(\xi)}{1+ib\kappa} = \Lambda^R + i\Lambda^I,$$
$$\frac{\mu(r,\omega)}{\lambda^*} = M(\xi,\kappa) = \frac{ib\kappa\varphi_{22}(\xi)+\varphi_{21}(\xi)}{1+ib\kappa} = M^R + iM^I.$$

Here: $\Lambda^R(\xi,\kappa) = \frac{\varphi_{11}(\xi)+b^2\kappa^2\varphi_{12}(\xi)}{1+b^2\kappa^2}$, $\Lambda^I(\xi,\kappa) = b\kappa\frac{\varphi_{12}(\xi)-\varphi_{11}(\xi)}{1+b^2\kappa^2}$; $M^R(\xi,\kappa) = \frac{\varphi_{21}(\xi)+b^2\kappa^2\varphi_{22}(\xi)}{1+b^2\kappa^2}$, $M^I(\xi,\kappa) = b\kappa\frac{\varphi_{22}(\xi)-\varphi_{21}(\xi)}{1+b^2\kappa^2}$; $\xi = \frac{r}{r_2}$ — dimensionless radial coordinate; parameter $b = \frac{n}{r_2}\sqrt{\frac{\lambda^*}{\rho}}$ is proportional to the relaxation time n, and parameter $\kappa^2 = \frac{\rho^*\omega^2 r_2^2}{\lambda^*}$ is proportional to the oscillation frequency; λ^* — dimensional parameter (characteristic value λ); ρ^* — characteristic density value.

Within the framework of the problem under consideration, the properties of the elastic material of the cylinder and the inhomogeneous viscoelastic material of the coating are described by the following characteristics, which are given in the form of piecewise continuous functions:

$$\Lambda(\xi,\kappa) = \begin{cases} \lambda_0, & \xi\in[\xi_0,\xi_*], \\ \Lambda^R(\xi,\kappa)+i\Lambda^I(\xi,\kappa), & \xi\in[\xi_*,1], \end{cases}$$
$$M(\xi,\kappa) = \begin{cases} \mu_0, & \xi\in[\xi_0,\xi_*], \\ M^R(\xi,\kappa)+iM^I(\xi,\kappa), & \xi\in[\xi_*,1], \end{cases}$$
$$\frac{\rho(r)}{\rho^*} = \begin{cases} \rho_0, & \xi\in[\xi_0,\xi_*], \\ \eta(\xi), & \xi\in[\xi_*,1], \end{cases}$$

where $\xi_0 = \frac{r_1}{r_2}$, $\xi_* = \frac{r_*}{r_2}$. Here functions Λ and M take constant real values on the elastic cylinder $\xi\in[\xi_0,\xi_*]$ and they are variable complex-valued for the coating material $\xi\in[\xi_*,1]$.

Taking into account the type of boundary conditions, the law of load change and the nature of the properties of the coating material, the solution can be constructed in the form [2]:

$$\frac{u_r}{r_2}=\sum_{k=0}^{\infty}\left(u_k^R(\xi)+iu_k^I(\xi)\right)\cos(\nu_k z),\qquad \frac{u_z}{r_2}=\sum_{k=0}^{\infty}\left(w_k^R(\xi)+iw_k^I(\xi)\right)\sin(\nu_k z),$$
$$\frac{\sigma_{rr}}{\lambda^*}=\sum_{k=0}^{\infty}\left(s_{1k}^R(\xi)+is_{1k}^I(\xi)\right)\cos(\nu_k z),\qquad \frac{\sigma_{zr}}{\lambda^*}=\sum_{k=0}^{\infty}\left(s_{2k}^R(\xi)+is_{2k}^I(\xi)\right)\sin(\nu_k z),\tag{1}$$
$$\frac{\sigma_{\varphi\varphi}}{\lambda^*}=\sum_{k=0}^{\infty}\left(s_{3k}^R(\xi)+is_{3k}^I(\xi)\right)\cos(\nu_k z),\qquad \frac{\sigma_{zz}}{\lambda^*}=-\sum_{k=0}^{\infty}\left(s_{4k}^R(\xi)+is_{4k}^I(\xi)\right)\cos(\nu_k z).$$

It should be noted that with such representations, the boundary conditions at the ends will be fulfilled identically. Using differential equations of motion that define relations, orthogonality properties of systems of functions $\{\cos(\nu_k z)\}$, $\{\sin(\nu_k z)\}$ and separating the real and imaginary parts, we can obtain a set of canonical systems that include first-order differential equations with variable coefficients and the corresponding boundary conditions:

– for $k=0$

$$s_{10}^{R'}=-\frac{s_{10}^R-s_{30}^R}{\xi}-\kappa^2\eta u_0^R,$$
$$s_{10}^{I'}=-\frac{s_{10}^I-s_{30}^I}{\xi}-\kappa^2\eta u_0^I,$$
$$u_0^{R'}=\frac{s_{10}^R A^R+s_{10}^I A^I-\frac{u_0^R}{\xi}\left(\Lambda^R A^R+\Lambda^I A^I\right)-\frac{u_0^I}{\xi}\left(\Lambda^R A^I-\Lambda^I A^R\right)}{A^2},$$
$$u_0^{I'}=-\frac{s_{10}^R A^I-s_{10}^I A^R-\frac{u_0^R}{\xi}\left(\Lambda^R A^I-\Lambda^I A^R\right)+\frac{u_0^I}{\xi}\left(\Lambda^R A^R+\Lambda^I A^I\right)}{A^2},$$
$$s_{10}^R(\xi_0)=0,\quad s_{10}^I(\xi_0)=0,$$
$$s_{10}^R(1)=p_0,\quad s_{10}^I(1)=0;$$

– for other values $k=1,2,\ldots$

$$s_{1k}^{R'}=-\beta_k s_{2k}^R-\frac{s_{1k}^R-s_{3k}^R}{\xi}-\kappa^2\eta u_k^R,$$
$$s_{1k}^{I'}=-\beta_k s_{2k}^I-\frac{s_{1k}^I-s_{3k}^I}{\xi}-\kappa^2\eta u_k^I,$$
$$s_{2k}^{R'}=s_{4k}^R-\frac{s_{2k}^R}{\xi}-\kappa^2\eta w_k^R,$$
$$s_{2k}^{I'}=s_{4k}^I-\frac{s_{2k}^I}{\xi}-\kappa^2\eta w_k^I,$$
$$u_k^{R'}=\frac{s_{1k}^R A^R+s_{1k}^I A^I-\left(\frac{u_k^R}{\xi}+\beta_k w_k^R\right)\left(\Lambda^R A^R+\Lambda^I A^I\right)-\left(\frac{u_k^I}{\xi}+\beta_k w_k^I\right)\left(\Lambda^R A^I-\Lambda^I A^R\right)}{A^2},$$
$$u_k^{I'}=-\frac{s_{1k}^R A^I-s_{1k}^I A^R-\left(\frac{u_k^R}{\xi}+\beta_k w_k^R\right)\left(\Lambda^R A^I-\Lambda^I A^R\right)+\left(\frac{u_k^I}{\xi}+\beta_k w_k^I\right)\left(\Lambda^R A^R+\Lambda^I A^I\right)}{A^2},$$

$$
\begin{aligned}
& w_k^{R'} = \frac{s_{2k}^R M^R + s_{2k}^I M^I}{\left(M^R\right)^2 + \left(M^I\right)^2} + \beta_k u_k^R, \\
& w_k^{I'} = -\frac{s_{2k}^R M^I + s_{2k}^I M^R}{\left(M^R\right)^2 + \left(M^I\right)^2} + \beta_k u_k^I, \\
& s_{1k}^R(\xi_0) = 0, \quad s_{1k}^I(\xi_0) = 0, \\
& s_{2k}^R(\xi_0) = 0, \quad s_{2k}^I(\xi_0) = 0, \\
& s_{1k}^R(1) = p_k, \quad s_{1k}^I(1) = 0, \\
& s_{2k}^R(1) = 0, \quad s_{2k}^I(1) = 0,
\end{aligned}
$$

where, for brevity and convenience of notation, the notation is introduced:

$$
\begin{gathered}
s_{3k}^R = A^R \frac{u_k^R}{\xi} + \Lambda^R\left(u_k^{R'} + \beta_k w_k^R\right) - A^I \frac{u_k^I}{\xi} - \Lambda^I\left(u_k^{I'} + \beta_k w_k^I\right), \\
s_{3k}^I = A^R \frac{u_k^I}{\xi} + \Lambda^R\left(u_k^{I'} + \beta_k w_k^I\right) + A^I \frac{u_k^R}{\xi} + \Lambda^I\left(u_k^{R'} + \beta_k w_k^R\right), \\
s_{4k}^R = A^R \beta_k^2 w_k^R + \Lambda^R\left(\beta_k u_k^{R'} + \frac{\beta_k u_k^R}{\xi}\right) - A^I \beta_k^2 w_k^I - \Lambda^I\left(\beta_k u_k^{I'} + \frac{\beta_k u_k^I}{\xi}\right), \\
s_{4k}^I = A^R \beta_k^2 w_k^I + \Lambda^R\left(\beta_k u_k^{I'} + \frac{\beta_k u_k^I}{\xi}\right) + A^I \beta_k^2 w_k^R + \Lambda^I\left(\beta_k u_k^{R'} + \frac{\beta_k u_k^R}{\xi}\right), \\
A^R = \Lambda^R + 2M^R, \qquad A^I = \Lambda^I + 2M^I, \qquad A^2 = \left(\Lambda^R + 2M^R\right)^2 + \left(\Lambda^I + 2M^I\right)^2, \\
\beta_k = \frac{\pi k r_2}{h}, \qquad p_k = -\frac{p^0 P_k}{\lambda^*}.
\end{gathered}
$$

In the general case of inhomogeneity of the coating material, the solutions to the formulated problems can only be obtained numerically, for example, by the shooting method. Following this method, we represent the functions u_0^R, u_0^I, for $k=0$ (zero harmonic) in the form:

$$
u_0^R = \alpha_1 u_0^{R(1)} + \alpha_2 u_0^{R(2)}, \qquad u_0^I = \alpha_1 u_0^{I(1)} + \alpha_2 u_0^{I(2)},
$$

where $u_0^{R(i)}$, $u_0^{I(i)}$ $(i=1,2)$ — solutions of the corresponding Cauchy problems with initial conditions:

$$
\left[\begin{array}{l}
s_{10}^{R(1)}(\xi_0) = 0, \\
s_{10}^{I(1)}(\xi_0) = 0, \\
u_0^{R(1)}(\xi_0) = 1, \\
u_0^{I(1)}(\xi_0) = 0,
\end{array}\right.
\qquad
\left[\begin{array}{l}
s_{10}^{R(2)}(\xi_0) = 0, \\
s_{10}^{I(2)}(\xi_0) = 0, \\
u_0^{R(2)}(\xi_0) = 0, \\
u_0^{I(2)}(\xi_0) = 1.
\end{array}\right.
$$

The real coefficients α_i are determined from the condition of satisfying the boundary conditions on the outer surface of the coating for the radial stress component:

$$
\begin{cases}
s_{10}^R(1) = \alpha_1 s_{10}^{R(1)}(1) + \alpha_2 s_{10}^{R(2)}(1) = p_0, \\
s_{10}^I(1) = \alpha_1 s_{10}^{I(1)}(1) + \alpha_2 s_{10}^{I(2)}(1) = 0.
\end{cases}
$$

Solutions of other problems (for $k=1,2,\ldots$) can also be found numerically from the representations:

$$u_k^R = \gamma_1 u_k^{R(1)} + \gamma_2 u_k^{R(2)} + \gamma_3 u_k^{R(3)} + \gamma_4 u_k^{R(4)}, \qquad u_k^I = \gamma_1 u_k^{I(1)} + \gamma_2 u_k^{I(2)} + \gamma_3 u_k^{I(3)} + \gamma_4 u_k^{I(4)},$$

where $u_0^{R(i)}$, $u_0^{I(i)}$ $(i=\overline{1,4})$ — solutions of Cauchy problems with initial conditions:

$$\left[\begin{array}{l} s_{1k}^{R(1)}(\xi_0)=0,\\ s_{1k}^{I(1)}(\xi_0)=0,\\ s_{2k}^{R(1)}(\xi_0)=0,\\ s_{2k}^{I(1)}(\xi_0)=0,\\ u_k^{R(1)}(\xi_0)=1,\\ u_k^{I(1)}(\xi_0)=0,\\ w_k^{R(1)}(\xi_0)=0,\\ w_k^{I(1)}(\xi_0)=0, \end{array}\right. \quad \left[\begin{array}{l} s_{1k}^{R(2)}(\xi_0)=0,\\ s_{1k}^{I(2)}(\xi_0)=0,\\ s_{2k}^{R(2)}(\xi_0)=0,\\ s_{2k}^{I(2)}(\xi_0)=0,\\ u_k^{R(2)}(\xi_0)=0,\\ u_k^{I(2)}(\xi_0)=1,\\ w_k^{R(2)}(\xi_0)=0,\\ w_k^{I(2)}(\xi_0)=0, \end{array}\right. \quad \left[\begin{array}{l} s_{1k}^{R(3)}(\xi_0)=0,\\ s_{1k}^{I(3)}(\xi_0)=0,\\ s_{2k}^{R(3)}(\xi_0)=0,\\ s_{2k}^{I(3)}(\xi_0)=0,\\ u_k^{R(3)}(\xi_0)=0,\\ u_k^{I(3)}(\xi_0)=0,\\ w_k^{R(3)}(\xi_0)=1,\\ w_k^{I(3)}(\xi_0)=0, \end{array}\right. \quad \left[\begin{array}{l} s_{1k}^{R(4)}(\xi_0)=0,\\ s_{1k}^{I(4)}(\xi_0)=0,\\ s_{2k}^{R(4)}(\xi_0)=0,\\ s_{2k}^{I(4)}(\xi_0)=0,\\ u_k^{R(4)}(\xi_0)=0,\\ u_k^{I(4)}(\xi_0)=0,\\ w_k^{R(4)}(\xi_0)=0,\\ w_k^{I(4)}(\xi_0)=1. \end{array}\right.$$

The values of the coefficients γ_i are established from the solution of the algebraic system, which follows from the condition of satisfaction of the boundary conditions for the components of normal and shear stresses:

$$\begin{aligned} s_{1k}^R(1) &= \gamma_1 s_{1k}^{R(1)}(1) + \gamma_2 s_{1k}^{R(2)}(1) + \gamma_3 s_{1k}^{R(3)}(1) + \gamma_4 s_{1k}^{R(4)}(1) = p_k,\\ s_{1k}^I(1) &= \gamma_1 s_{1k}^{I(1)}(1) + \gamma_2 s_{1k}^{I(2)}(1) + \gamma_3 s_{1k}^{I(3)}(1) + \gamma_4 s_{1k}^{I(4)}(1) = 0,\\ s_{2k}^R(1) &= \gamma_1 s_{2k}^{R(1)}(1) + \gamma_2 s_{2k}^{R(2)}(1) + \gamma_3 s_{2k}^{R(3)}(1) + \gamma_4 s_{2k}^{R(4)}(1) = 0,\\ s_{2k}^I(1) &= \gamma_1 s_{2k}^{I(1)}(1) + \gamma_2 s_{2k}^{I(2)}(1) + \gamma_3 s_{2k}^{I(3)}(1) + \gamma_4 s_{2k}^{I(4)}(1) = 0. \end{aligned}$$

With the help of the described procedure, it is possible to construct a numerical solution of the original problem for the «cylinder–coating» system under the given laws of change in the properties of an inhomogeneous viscoelastic coating and keeping a finite number of terms in representations (1). It should be noted that in the right parts of the differential equations for problems for $k=0,1,2,\ldots$ there are no derivatives of functions Λ^R, Λ^I, M^R, M^I. Due to this, when modeling oscillations of a cylinder with a viscoelastic coating, it is possible to use a wide class of laws for changing the properties of the coating material (monotonic, nonmonotonic, piecewise constant).

The basis of another numerical approach, which can be used to study the problem under consideration, is the FEM. The package that implements this method is FlexPDE. Its main advantages are the ability to explicitly specify oscillation equations and constitutive relations; the presence of built-in functionality for working with complex values; easy installation on a computer and low requirements for its technical characteristics.

The oscillation equations and boundary conditions, which correspond to the dimensionless formulation of the problem and are implemented in the FlexPDE software environment, have the form:

$$\begin{aligned} &S_{11,\xi} + S_{12,x} + \frac{S_{11}-S_{22}}{\xi} + \kappa^2 u_1^* = 0,\\ &S_{13,\xi} + S_{33,x} + \frac{S_{13}}{\xi} + \kappa^2 u_3^* = 0,\\ &u_3^* = 0, \quad S_{13} = 0, \quad x = \pm h^*, \quad \xi \in [\xi_0, 1],\\ &S_{11} = 0, \quad S_{13} = 0, \quad \xi = \xi_0, \quad x \in [-h^*, h^*],\\ &S_{11} = p, \quad S_{13} = 0, \quad \xi = 1, \quad x \in [-h^*, h^*], \end{aligned} \tag{2}$$

where, by analogy with the first approach, dimensionless quantities and functions are introduced:

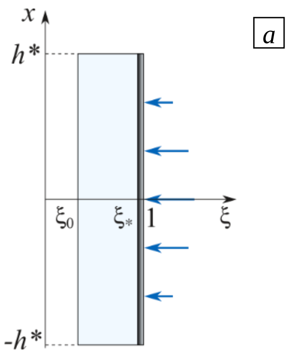
$$S_{11} = \sigma_{rr}/\lambda^*, \qquad S_{22} = \sigma_{\varphi\varphi}/\lambda^*, \qquad S_{13} = \sigma_{zr}/\lambda^*, \qquad S_{33} = \sigma_{zz}/\lambda^*,$$

$$u_1^* = u_r/r_2\,, \qquad u_3^* = u_z/r_2\,, \qquad x = z/r_2 \in \left[-h^*, h^*\right], \qquad h^* = h/r_2\,.$$

The constitutive relations for dimensionless stresses are given explicitly:

$$S_{11} = \Lambda\vartheta + 2M\varepsilon_{11}\,, \qquad S_{13} = 2M\varepsilon_{13}\,, \qquad S_{22} = \Lambda\vartheta + 2M\varepsilon_{22}\,, \qquad S_{33} = \Lambda\vartheta + 2M\varepsilon_{33}\,,$$
$$\varepsilon_{11} = \frac{\partial u_1^*}{\partial \xi}\,, \qquad \varepsilon_{22} = \frac{u_1^*}{\xi}\,, \qquad \varepsilon_{33} = \frac{\partial u_3^*}{\partial x}\,, \qquad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_3^*}{\partial \xi} + \frac{\partial u_1^*}{\partial x}\right), \qquad \vartheta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\,.$$

Taking into account the symmetry of the problem, we construct its solution on the area, which is half of the longitudinal section of the coated cylinder (Fig. 2).



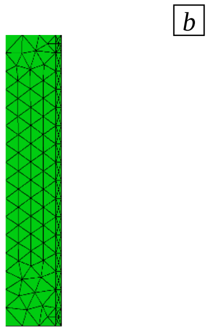


Fig. 2. Part of the longitudinal section of the «cylinder-coating» system (*a*) and an example of the computational grid (*b*).

The formulated dimensionless statement makes it possible to carry out unified calculations for coated cylinders with different geometric dimensions but with the same ratio of half-height to the outer radius. It should be noted that a distinctive feature of the approach based on the FEM, in comparison with the first approach, is the possibility of a fairly simple solution of the problem for various types of boundary conditions.

Thus, using the two approaches described, it is possible to carry out a numerical study of the influence of the properties of the coating material on the amplitude-frequency characteristics (AFC) of the «cylinder-coating» system, build graphs of the change in the components of the stress tensor and the displacement field depending on the geometric and physical-mechanical parameters of the problem.

3. Computational experiments

To evaluate the accuracy of the numerical methods used and to analyze the effect of the viscoelastic properties of the coating on the values of the displacement function components, let us consider, as an example, a cylinder with geometric parameters $h = 0{,}5$ m, $r_1 = 0{,}8$ m, $r_* = 0{,}98$ m, $r_2 = 1$ m that correspond to dimensionless quantities: $h^* = 0{,}5$, $\xi_0 = 0{,}8$, $\xi_* = 0{,}98$. Here the coating thickness is 10% of the cylinder thickness. Let us define an even function of load change in the form of a parabola: $q(z) = 1 - (z/h)^2$. As a cylinder material, we take aluminum, which has the following physical and mechanical characteristics: Young's modulus $E_{Al} = 70$ GPa, Poisson's ratio $\nu_{Al} = 0{,}34$, and Lame parameters (up to the second decimal place) $\lambda_{Al} = \dfrac{E_{Al}\nu_{Al}}{(1+\nu_{Al})(1-2\nu_{Al})} = \lambda_0\lambda^* = 55{,}5$ GPa, $\mu_{Al} = \dfrac{E_{Al}}{2(1+\nu_{Al})} = \mu_0\lambda^* = 26{,}12$ GPa. As a coating material, we consider an inhomogeneous polyvinyl chloride with a relaxation time $n = 10^{-4}$ s, Poisson's ratio at small deformations $\nu_{PVCi} = 0{,}38$, and an instantaneous modulus of elasticity $E_{PVCi} = 2{,}7$ GPa at the interface ($\xi = \xi_*$). We set the dimensional parameter λ^* to be unit: $\lambda^* = 1$ GPa. Since the influence of only the viscoelastic properties of the coating is being studied, we will assume that the density of its material is constant ($\eta(\xi) = 1$) and equal to the density of aluminum $\rho^* = 2712$ kg/m^3. Then the dimensionless parameter takes the value: $b = 0{,}06$. Taking into account the fact that the long-term modulus is greater than the instantaneous one, we choose the following functions φ_{ij}:

$$\varphi_{11}(\xi)=1{,}63+1{,}5\xi\,,\quad \varphi_{21}(\xi)=0{,}19+0{,}81\xi\,,\quad \varphi_{12}(\xi)=1{,}2(1{,}63+1{,}5\xi)\,,\quad \varphi_{22}(\xi)=1{,}2(0{,}19+0{,}81\xi)\,. \tag{3}$$

Figures 3, 4 show graphs of real and imaginary dimensionless functions of change in the components of the radial displacement $u_1^*(\xi)$ and stress $S_{11}(\xi)$, built according to the solution of problem (2) using the FEM for $x=0$, and graphs of the corresponding functions obtained according to the first approach from (1) with a different number of terms held N.

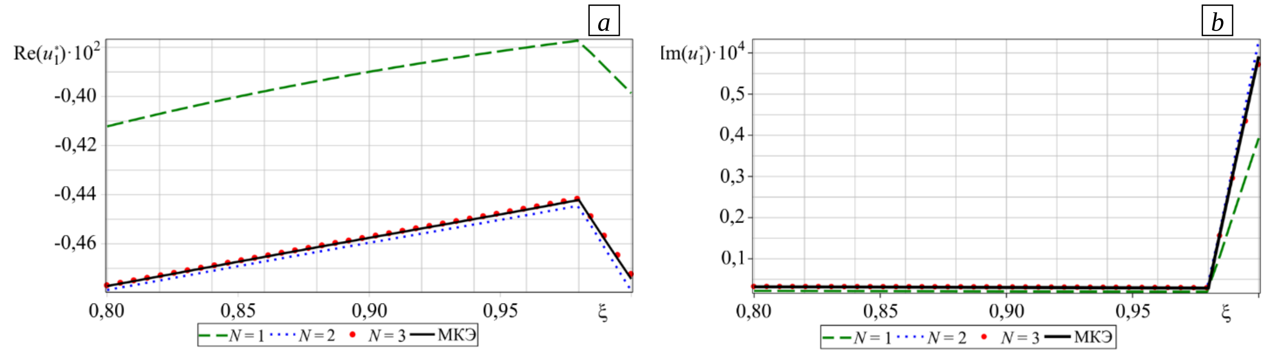


Fig. 3. Dependences $\mathrm{Re}(u_1^*)$ (*a*) and $\mathrm{Im}(u_1^*)$ (*b*) on ξ.

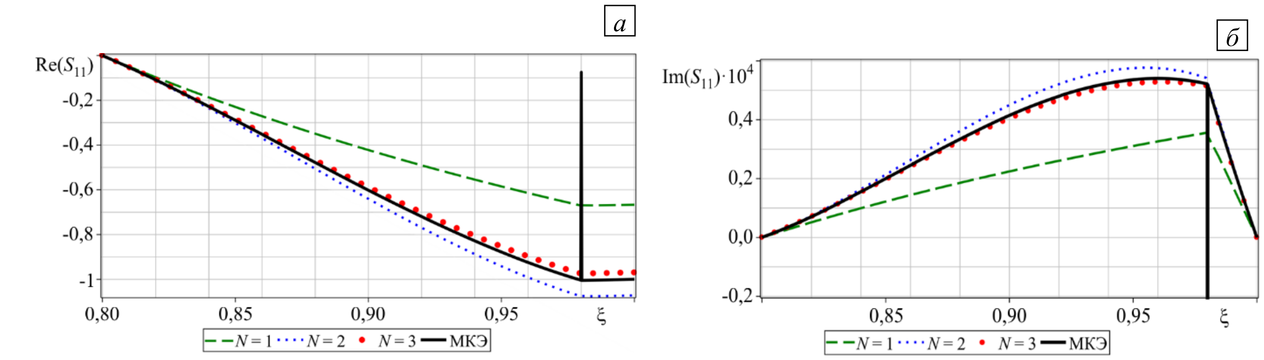


Fig. 4. Dependences $\mathrm{Re}(S_{11})$ (*a*) and $\mathrm{Im}(S_{11})$ (*b*) on ξ.

The frequency parameter takes the values: $\kappa=1{,}2$. As can be seen, there is a convergence of solutions. There is also a feature in the graphs of the function $S_{11}(\xi)$ near the point $\xi=\xi_*$, corresponding to the FEM solution. This is due to the error in the numerical determination of the values of the derivatives of the components of the displacement functions at this point.

To evaluate the convergence of the numerical solution implemented in the FlexPDE package, the function values $\mathrm{Re}(u_1^*)$ were obtained at three points: $(\xi_0,0)$, $(\xi_*,0)$, $(1,0)$, for a different number of adaptive grid nodes: 56, 102, 239. The number of nodes was determined automatically according to the parameter values specified in the package ngrid $=1$, 14, 28, which characterizes the density of the generated mesh. The calculation results are presented in the table 1.

Table 1. Function values $\mathrm{Re}(u_1^*)$ calculated at three points for different finite element meshes

Number of mesh nodes	$\mathrm{Re}(u_1^*)$		
	$(\xi_0,0)$	$(\xi_*,0)$	$(1,0)$
56	–0,0477131726	–0,0442096610	–0,0474257445
102	–0,0477176110	–0,0442117023	–0,0474274165
250	–0,0477181887	–0,0442122318	–0,0474277848

Using the methods used, it is possible to analyze the influence of the laws of inhomogeneity and parameters of the coating material on the AFC and the displacement field of a piecewise homogeneous system. For example, Figure 5a shows plots $\left|u_1^*\right|$ at the point $(1,0)$ for $\kappa\in[9,54;\,9,56]$ and with different values of the parameter $b=0,06;\,0,08$. The following laws are considered as other laws of change in properties:

$$\varphi_{11}(\xi)=1,63+1,5\xi^2,\quad \varphi_{21}(\xi)=0,19+0,81\xi^2,\quad \varphi_{12}(\xi)=1,2\left(1,63+1,5\xi^2\right),$$
$$\varphi_{22}(\xi)=1,2\left(0,19+0,81\xi^2\right). \tag{4}$$

Figure 5*b* contains plots of the function $\mathrm{Re}\left(u_1^*\right)$, corresponding to the laws (3) and (4) for $b=0,06$ and

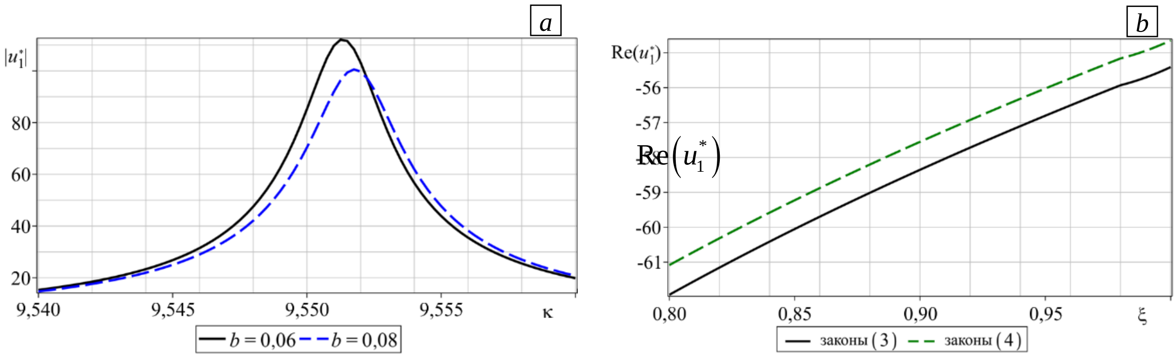


Fig. 5. Plots $\left|u_1^*\right|$, corresponding different parameter values b (*a*); plots $\mathrm{Re}\left(u_1^*\right)$ for $b=0,06$ and $\kappa=9,55$ laws (3) and (4) (*b*).

$\kappa=9,55$. It should be noted that the curves differ from each other for frequencies close to the frequency corresponding to the maximum value $\left|u_1^*\right|$.

The graphs show that there is a significant influence of the properties of the coating material on AFC of the «cylinder-coating» system. Such data can be used in the study of oscillatory processes corresponding to inverse coefficient problems [8, 9].

4. Conclusion

To solve the problem of steady longitudinal-radial vibrations of a cylinder with a viscoelastic coating, two approaches based on numerical methods were applied. The obtained results are compared. As the advantages of the approach based on representation (1), one can note, firstly, the simplicity of its implementation in programs written in modern programming languages C++, Python, C# and others, or in the mathematical packages Maple, MATLAB, Mathematica, and secondly, the absence of singularities in the solution in the vicinity of the boundary between the cover and the cylinder. It can be said about the approach based on the FEM from the FlexPDE package that it allows one to build a solution not only for materials whose properties change only along the radial coordinate, but also for two-dimensional inhomogeneity laws. With the help of FEM, it is also possible to study various types of fastenings and loadings.

Both approaches make it possible to study the influence of the variable properties of the material of the object on the frequency response, the values of the components of the displacement and stress fields, to simulate the stress-strain state of inhomogeneous isotropic elastic and viscoelastic pipes, the properties of which are described by the variable Lamé parameters and density. As a result, at the design stage of modern pipes (metal-plastic, polymeric materials, in particular, polyvinyl chloride, high-density or low-density polyethylene), the choice of the optimal material becomes available. In addition, the revealed fact of the influence of the variable properties of the material of the object on its frequency response and displacement fields can be useful in solving inverse coefficient problems of determining the laws of change in these properties [8, 9].

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