COMPUTATIONAL CONTINUUM MECHANICS
ВЫЧИСЛИТЕЛЬНАЯ МЕХАНИКА СПЛОШНЫХ СРЕД
Computational Continuum Mechanics. 2021. Vol. 14. No. 3. pp. 253-263
DOI: 10.7242/1999-6691/2021.14.3.21

SOLUTION OF THE GRADIENT THERMOELASTICITY PROBLEM FOR A CYLINDER WITH A HEAT-PROTECTED COATING

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The study of the stress-strain state of an infinitely long thermoelastic cylinder has been carried out taking into account scale effects. A thermal protective coating is applied to the outer side surface of the cylinder, the thermomechanical characteristics of which are functions of the radial coordinate. Thermal boundary conditions of the 1st kind are set on stress-free side surfaces of the cylinder. To account for scale effects, the Aifantis' one-parameter gradient theory of thermoelasticity is used. Additional boundary conditions and conjugation conditions for couple stresses are specified. The displacements and stresses are represented as the sum of the solutions of the thermoelasticity problem in the classical formulation and the gradient parts. After finding the radial temperature distribution, the thermoelasticity problem in the classical formulation with respect to radial displacements and stresses is solved numerically by the shooting method. Additional boundary layer terms for radial displacements at small values of the gradient parameter are found using the asymptotic method for solving linear differential equations with spatially varying coefficients (the Wentzel–Kramers–Brillouin method – WKB method). Specific examples of calculations are given for radial displacements, Cauchy stresses, couple stresses and total stresses in the case of both homogeneous and inhomogeneous coatings. It has been found that Cauchy stresses and total stresses experience a jump at the boundary between the cylinder and the coating. The couple stresses at small gradient parameters are much less than the total stresses. An increase of the scale parameter reduces the values of radial displacements and total stresses. Deformations are continuous in the cases of both homogeneous and inhomogeneous coating. A comparative analysis of the influence of the inhomogeneity parameter value on the distribution of displacements and total stresses has been done.

***Key words**:* gradient thermoelasticity, hollow cylinder, Cauchy problems, shooting method, WKB method, thermal protection coating, inhomogeneous materials

1.Introduction

Thermal protective coatings are designed to protect against premature destruction of structural elements of constructions operating in a high-temperature environment. The main feature of thermal protective coatings is the low coefficient of thermal conductivity, as a result of which the temperature of the metal substrate becomes much lower than the temperature on the surface of the coating. Coatings are usually made from homogeneous materials such as ceramics. However, due to the difference in the thermomechanical properties of the coating and the substrate, areas of stress concentration can appear on their interface, which can lead to delamination of the coating. As an alternative to homogeneous and layered coatings in recent years, inhomogeneous coatings have come forward, in which the thermomechanical properties are not constants, but coordinate functions [1]. When they are used, there are no jumps in material properties at the "coating-substrate" interface.

The most common method for studying the state of inhomogeneous bodies is the finite element method [2]. Other approximate methods are also used. For example, in [3], an economical numerical-analytical method for solving the thermoelasticity problem for an elongated rectangle consisting of two types of heat-shielding coatings, homogeneous and functionally graded, is proposed. For each type of coating, its own model of thermoelastic deformation of the "coating-substrate" system is presented. The deformation models are based on special distribution laws for the components of the displacement vector and temperature over the coating thickness, which made it possible to take into account the effect of inhomogeneity of thermomechanical characteristics and satisfy the conditions for matching displacement fields, stresses, temperature and heat flow. The unknown functions included in the expressions for the temperature and displacement transformants were determined using the variational principle of thermoelasticity and the Kantorovich method.

When calculating the stress state in microsized objects, the gradient theory of elasticity is used. This theory takes into account scale effects, that is, the dependence of the stress-strain state (SSS) on the characteristic dimensional parameters. The gradient theory of elasticity, formulated in the middle of the last century in the works of Tupin [4] and Mindlin [5], later received its generalization to the theory of thermoelasticity [6].

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For the first time, the Lame problem from the standpoint of the gradient theory of elasticity was considered in [7] for an infinite-length pipe and a spherical shell, which are under the action of uniform external and internal pressure. However, the practical use of the Tupin–Mindlin model ran into the question of identifying five additional gradient modules. In order to simplify the constitutive equations, the applied gradient models of Aifantis [8–11] and Lurie [12] were proposed, the constitutive equations of which include only one gradient parameter. Within the framework of the Aifantis one-parameter model, in [13], on the basis of the variational principle, the equilibrium equations and boundary conditions of the problem of equilibrium of a homogeneous thick-walled hollow cylinder under mechanical loading were obtained, and an analytical solution of the problem was found using the apparatus of modified Bessel functions. Analytical solutions have also been obtained for the problem of determining the SSS of an inhomogeneous hollow cylinder under power laws of inhomogeneity of thermal and mechanical loading [14–16].

Single-parameter gradient models of mechanics are also used to refine the SSS of layered elastic [17–20] and thermoelastic [21, 22] bodies. In [17], in the framework of the interfacial layer model, S.A. Lurie numerically investigates the equilibrium of a layer with a coating under the influence of a localized normal load. The solution is obtained using the integral Fourier transform and its numerical inversion. In [20], the problem of bending a microbeam with a partial coating is solved. The effect of a change in the magnitude of the scale parameter on the nature of the distribution of displacements and stresses in the beam and on the position of its neutral line is studied. Under the assumption that the problem is one-dimensional, the SSS of thin-layer composite structures under thermal action was analytically studied in [21]. The authors of [22] solved the problem of gradient thermoelasticity for a composite rod, and to find the Cauchy stresses, the asymptotic Vishik–Lyusternik approach was used, which takes into account the presence of boundary layer solutions in the vicinity of the boundaries and the conjugation point of the rods. The dependence of the Cauchy stress jump on the ratio of the physical characteristics of the rods and the scale parameter is also studied.

The objectives of this work are as follows: formulation of the problem of gradient thermoelasticity for a hollow cylinder with a thermal protective coating under effective nondimensionalization; selection, on the basis of the asymptotic Wentzel–Kramers–Brillouin method, the WKB method, of the boundary layer parts of the solution; using the shooting method to solve the problem of thermoelasticity in the classical formulation; testing the solution method on the example of a homogeneous coating; calculation of radial displacements, Cauchy stresses, total and couple stresses for both homogeneous and inhomogeneous coatings; the analysis of the obtained results.

2. Constitutive relations of gradient mechanics

In the gradient theory of elasticity, the strain energy density depends not only on the strain, but also on its first gradient [5]. In the case of a linear isotropic material, the expression for the strain energy density in the framework of the Aifantis one-parameter model has the form [8]:

$$w=(\lambda/2)\varepsilon_{ii}\varepsilon_{jj}+\mu\varepsilon_{ij}\varepsilon_{ij}+l^2\left((\lambda/2)\varepsilon_{ii,k}\varepsilon_{jj,k}+\mu\varepsilon_{ij,k}\varepsilon_{ij,k}\right), \tag{1}$$

where λ and μ are the Lamé parameters, ε_{ij} are the components of the strain tensor, l is the gradient parameter that characterizes the microstructural structure of the material and has the dimension of length (for example, for polycrystalline solids, this is the grain size).

In the gradient theory of elasticity, the components of the Cauchy stress tensor τ_{ij}, couple stress tensor m_{ijs}, and total stress tensor σ_{ij} are represented as:

$$\tau_{ij}=\frac{\partial w}{\partial\varepsilon_{ij}}, \tag{2}$$

$$m_{ijs}=\frac{\partial w}{\partial\varepsilon_{ij,s}}=l^2\tau_{ij,s}, \tag{3}$$

$$\sigma_{ij}=\tau_{ij}-m_{ijs,s}=\left(1-l^2\nabla^2\right)\tau_{ij}. \tag{4}$$

The mathematical formulation of the problem of gradient thermoelasticity includes [6]:
– the equilibrium equations written in total stresses

$$\sigma_{ij,j} = 0\,; \tag{5}$$

– mechanical static boundary conditions

$$\tau_{ij} n_j - m_{ijs,s} n_j - \left(m_{ijs,s} n_s\right)_{,j} + \left(m_{ijs,s} n_j n_s\right)_{,z} n_z = t_i\,, \qquad m_{ijs,s} n_j n_s = \rho_i\,; \tag{6}$$

– mechanical kinematic boundary conditions

$$u_i = v_i\,, \qquad u_{i,j}\, n_j = \frac{\partial v_i}{\partial n}\,; \tag{7}$$

– classical heat equation

$$\left(k_{ij} T_{,i}\right)_{,j} = 0\,; \tag{8}$$

– thermal boundary conditions of the 1st kind, given on the surface S_T

$$T\big|_{S_T} = T_0\,; \tag{9}$$

– thermal boundary conditions of the 2nd kind, given on the surface S_q

$$q\big|_{S_q} = q_0\,. \tag{10}$$

Here: t_i, ρ_i are the components of force vectors given on the surface of the body; n_i are the components of the unit vector of the normal to the body surface at the considered point; T is a temperature; k_{ij} are the components of the thermal conductivity tensor; q is a heat flow; $S = S_q \bigcup S_T$ is the surface of the body; ∇ is the nabla operator (Hamilton differential operator); the comma in the subscript means differentiation with respect to the coordinate.

In the polar coordinate system, the expressions for the nonzero components of the tensors (2)–(4) and the components of the vectors (6) have the form [14, 15]:

$$\tau_{rr} = (\lambda + 2\mu)\frac{du_r}{dr} + \lambda\frac{u_r}{r} - \gamma T\,, \qquad \tau_{\varphi\varphi} = (\lambda + 2\mu)\frac{u_r}{r} + \lambda\frac{du_r}{dr} - \gamma T\,,$$

$$m_{rrr} = l^2\frac{d\tau_{rr}}{dr}\,, \qquad m_{\varphi\varphi r} = l^2\frac{d\tau_{\varphi\varphi}}{dr}\,, \qquad m_{r\varphi\varphi} = m_{\varphi r\varphi} = \frac{1}{2}l^2\frac{\left(\tau_{rr} - \tau_{\varphi\varphi}\right)}{r}\,,$$

$$\sigma_{rr} = \tau_{rr} - \frac{1}{r}\frac{d}{dr}\left(r m_{rrr}\right) + \frac{1}{r}\left(m_{\varphi r\varphi} + m_{r\varphi\varphi}\right), \qquad \sigma_{\varphi\varphi} = \tau_{\varphi\varphi} - \frac{1}{r}\frac{d}{dr}\left(r m_{\varphi\varphi r}\right) - \frac{1}{r}\left(m_{\varphi r\varphi} + m_{r\varphi\varphi}\right),$$

$$t_r = \tau_{rr} - \frac{1}{r}\frac{d}{dr}\left(r m_{rrr}\right) + \frac{1}{r}\left(m_{\varphi\varphi r} + m_{\varphi r\varphi} + m_{r\varphi\varphi}\right), \qquad \rho_r = m_{rrr}\,.$$

Here γ is the thermal stress coefficient.

3. Statement of the gradient thermoelasticity problem for a cylinder

Let us consider an infinitely long thermoelastic cylinder, on the outer side surface $r = h_1$ of which a thermal protective coating with a thickness h is applied. Zero temperature is maintained on the inner side surface $r = a$ of the cylinder. The outer surface of the coating $r = b$, where $b = h_1 + h$, is stress free and is at a temperature of T_0. The material of the cylinder is homogeneous, characterized by Lame coefficients λ_1 and μ_1, thermal conductivity coefficient k_1, thermal stress coefficient γ_1. The coating material has characteristics λ_2, μ_2, k_2, γ_2, depending on the coordinate r. Let us represent the material characteristics of the "cylinder-coating" system in the form of piecewise continuous functions:

$$F(r)=\begin{cases} F_1=\text{const}, & r\in[a,b], \\ F_2(r), & r\in(b,c], \end{cases} \tag{11}$$

where any of the material characteristics of the cylinder (or coating) can act as F_1 (or F_2).

According to the classical formulation of the problem, the conditions of continuity in temperature, heat flux, displacements, and radial stresses must be satisfied on the mating surface of the coating and the cylinder $r=h_1$. Since the equilibrium equations in the gradient theory have an increased order of differential equations compared to the classical theory, we will set additional boundary conditions and conjugation conditions. As additional conditions, we assume: 1) equality to zero of couple stresses on the side surfaces of the cylinder; 2) continuity of displacement gradients and couple stresses on the mating surface of the cylinder and the coating $r=h_1$ [13]. To simplify the calculations, we will assume that the gradient parameter is the same for the coating and the cylinder, i.e. $l_1=l_2=l$. Thus, the formulation of the boundary value problem of thermoelastic deformation of a coated cylinder takes the form:

$$\frac{d\sigma_{rr}}{dr}+\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r}=0, \qquad r\in[a,b], \tag{12}$$

$$\sigma_{rr}=\tau_{rr}-l^2\left(\frac{d^2\tau_{rr}}{dr^2}+\frac{1}{r}\frac{d\tau_{rr}}{dr}-2\frac{(\tau_{rr}-\tau_{\varphi\varphi})}{r^2}\right), \tag{13}$$

$$\sigma_{\varphi\varphi}=\tau_{\varphi\varphi}-l^2\left(\frac{d^2\tau_{\varphi\varphi}}{dr^2}+\frac{1}{r}\frac{d\tau_{\varphi\varphi}}{dr}-2\frac{(\tau_{\varphi\varphi}-\tau_{rr})}{r^2}\right), \tag{14}$$

$$\frac{1}{r}\frac{d}{dr}\left(rk(r)\frac{dT}{dr}\right)=0, \qquad r\in[a,b], \tag{15}$$

$$T^{(1)}(a)=0, \quad T^{(2)}(b)=T_0, \quad T^{(1)}(h_1)=T^{(2)}(h_1), \quad k_1(h_1)\frac{\partial T^{(1)}}{\partial r}(h_1)=k_2(h_1)\frac{\partial T^{(2)}}{\partial r}(h_1), \tag{16}$$

$$t_r^{(1)}(a)=0, \qquad t_r^{(2)}(b)=0, \tag{17}$$

$$\rho_r^{(1)}(a)=0, \qquad \rho_r^{(2)}(b)=0, \tag{18}$$

$$u^{(1)}(h_1)=u^{(2)}(h_1), \quad \frac{du^{(1)}}{dr}(h_1)=\frac{du^{(2)}}{dr}(h_1), \quad t_r^{(1)}(h_1)=t_r^{(2)}(h_1), \quad \rho_r^{(1)}(h_1)=\rho_r^{(2)}(h_1). \tag{19}$$

Let us state dimensionless problem (12)–(19) by the formulas:

$$\xi=\frac{r}{b}, \quad U=\frac{u_r}{b}, \quad \Omega_{rr}=\frac{\sigma_{rr}}{\mu_0}, \quad \Omega_{\varphi\varphi}=\frac{\sigma_{\varphi\varphi}}{\mu_0}, \quad S_{rr}=\frac{\tau_{rr}}{\mu_0}, \quad S_{\varphi\varphi}=\frac{\tau_{\varphi\varphi}}{\mu_0},$$

$$M_{rrr}=\frac{m_{rrr}}{\mu_0 b}, \quad M_{\varphi\varphi r}=\frac{m_{\varphi\varphi r}}{\mu_0 b}, \quad M_{r\varphi\varphi}=\frac{m_{r\varphi\varphi}}{\mu_0 b}, \quad \bar{t}_r=\frac{t_r}{\mu_0}, \quad \bar{\rho}_r=\frac{\rho_r}{\mu_0 b}, \quad W=\frac{T}{T_0}, \quad \bar{k}=\frac{k}{k_0}, \quad \bar{\lambda}=\frac{\lambda}{\mu_0},$$

$$\bar{\mu}=\frac{\mu}{\mu_0},$$

$$\bar{\gamma}=\frac{\gamma}{\gamma_0}, \quad R_0=\frac{h_1}{b}, \quad \beta_0=\frac{\gamma_0 T_0}{\mu_0}, \quad \alpha=\frac{l}{b}, \quad \bar{a}=\frac{a}{b}, \quad k_0=\max_{r\in[a,b]}k(r), \quad \mu_0=\max_{r\in[a,b]}\mu(r), \quad \gamma_0=\max_{r\in[a,b]}\gamma(r).$$

In dimensionless form, the problem (12)–(19) will be as follows:

$$\frac{d\Omega_{rr}}{d\xi}+\frac{\Omega_{rr}-\Omega_{\varphi\varphi}}{\xi}=0, \qquad \xi\in\left[\bar{a},1\right], \tag{20}$$

$$\Omega_{rr}=S_{rr}-\alpha^2\left(\frac{d^2S_{rr}}{d\xi^2}+\frac{1}{\xi}\frac{dS_{rr}}{d\xi}-2\frac{\left(S_{rr}-S_{\varphi\varphi}\right)}{\xi^2}\right), \tag{21}$$

$$\Omega_{\varphi\varphi}=S_{\varphi\varphi}-\alpha^2\left(\frac{d^2S_{\varphi\varphi}}{d\xi^2}+\frac{1}{\xi}\frac{dS_{\varphi\varphi}}{dr}-2\frac{\left(S_{\varphi\varphi}-S_{rr}\right)}{\xi^2}\right), \tag{22}$$

$$\frac{d}{d\xi}\left(\xi\bar{k}\left(\xi\right)\frac{dW}{d\xi}\right)=0, \qquad \xi\in\left[\bar{a},1\right], \tag{23}$$

$$W^{(1)}\left(\bar{a}\right)=0, \qquad W^{(2)}\left(1\right)=1, \qquad W^{(1)}\left(R_0\right)=W^{(2)}\left(R_0\right), \qquad k_1\left(R_0\right)\frac{\partial W^{(1)}}{\partial\xi}\left(R_0\right)=k_2\left(R_0\right)\frac{\partial W^{(2)}}{\partial\xi}\left(R_0\right), \tag{24}$$

$$\bar{t}_r^{(1)}\left(\bar{a}\right)=0, \qquad \bar{t}_r^{(2)}\left(1\right)=0, \tag{25}$$

$$\bar{\rho}_r^{(1)}\left(\bar{a}\right)=0, \qquad \bar{\rho}_r^{(2)}\left(1\right)=0, \tag{26}$$

$$U^{(1)}\left(R_0\right)=U^{(2)}\left(R_0\right), \qquad \frac{dU^{(1)}}{d\xi}\left(R_0\right)=\frac{dU^{(2)}}{d\xi}\left(R_0\right), \qquad \bar{t}_r^{(1)}\left(R_0\right)=\bar{t}_r^{(2)}\left(R_0\right), \qquad \bar{\rho}_r^{(1)}\left(R_0\right)=\bar{\rho}_r^{(2)}\left(R_0\right). \tag{27}$$

In this case, the dimensionless Cauchy stresses will be written as

$$S_{rr}=\left(\bar{\lambda}+2\bar{\mu}\right)\frac{dU}{d\xi}+\bar{\lambda}\frac{U}{\xi}-\beta_0\bar{\gamma}W, \qquad S_{\varphi\varphi}=\left(\bar{\lambda}+2\bar{\mu}\right)\frac{U}{\xi}+\bar{\lambda}\frac{dU}{d\xi}-\beta_0\bar{\gamma}W. \tag{28}$$

4. Problem solution

Let us start solving the problem of uncoupled thermoelasticity (20)–(27) by finding the radial temperature distribution in the cylinder and in the coating. It follows from the solution of the heat conduction problem in the classical formulation (23), (24) and has the form:

$$W^{(1)}\left(\xi\right)=C_1f_1\left(\xi\right), \qquad W^{(2)}\left(\xi\right)=C_2f_2\left(\xi\right)+C_3. \tag{29}$$

where $f_1\left(\xi\right)=\int_{\bar{a}}^{\xi}\frac{d\eta}{\eta\bar{k}_1\left(\eta\right)}$, $f_2\left(\xi\right)=\int_{R_0}^{\xi}\frac{d\eta}{\eta\bar{k}_2\left(\eta\right)}$, $C_1=C_2=\frac{1}{f_1\left(R_0\right)+f_2\left(1\right)}$, $C_3=\frac{f_1\left(R_0\right)}{f_1\left(R_0\right)+f_2\left(1\right)}$.

After determining the temperature, we proceed to finding the displacements and stresses, which, according to [8], can be represented as the sum of the solution of the thermoelasticity problem in the classical formulation and additional gradient terms: $U=U_{clas}+U_{grad}$.

Assuming $\alpha=0$ in (20)–(27), we arrive at the classical statement of the problem, which in the case of an inhomogeneous coating material can only be solved numerically. Let us use the shooting method [23] for this. After some transformations, we obtain a canonical system of two inhomogeneous ordinary differential equations of the 1st order with respect to radial displacements U_{clas} and radial stresses Ω_{clas}:

$$U'_{clas}=\frac{1}{\bar{\lambda}+2\bar{\mu}}\Omega_{clas}-\frac{\bar{\lambda}}{\left(\bar{\lambda}+2\bar{\mu}\right)\xi}U_{clas}+\frac{\bar{\gamma}}{\bar{\lambda}+2\bar{\mu}}\beta_0W, \tag{30}$$

$$\Omega'_{clas}=\frac{1}{\xi}\left(\frac{\bar{\lambda}}{\bar{\lambda}+2\bar{\mu}}-1\right)\Omega_{clas}+\frac{1}{\xi^2}\left(\bar{\lambda}+2\bar{\mu}-\frac{\bar{\lambda}^2}{\bar{\lambda}+2\bar{\mu}}\right)U_{clas}+\frac{\bar{\gamma}}{\xi}\left(\frac{\bar{\lambda}}{\bar{\lambda}+2\bar{\mu}}-1\right)\beta_0W, \tag{31}$$

and boundary conditions:

$$\Omega_{clas}\left(\bar{a}\right)=0, \qquad \Omega_{clas}\left(1\right)=0. \tag{32}$$

According to the shooting method, the solution of the boundary value problem (30)–(32) described by the inhomogeneous system of ordinary differential equations can be represented as the sum of the solution of the inhomogeneous Cauchy problem (30), (31) with zero initial conditions on the inner surface: $U_0^{(1)}(\bar{a})=0$, $\Omega_0^{(1)}(\bar{a})=0$, and solutions homogeneous Cauchy problem (30), (31) with non-zero initial conditions on the inner surface: $U_1^{(1)}(\bar{a})=1$, $\Omega_1^{(1)}(\bar{a})=0$, multiplied by the coefficient p:

$$\Omega_{clas}(\xi)=\Omega_0(\xi)+p\Omega_1(\xi), \qquad U_{clas}(\xi)=U_0(\xi)+pU_1(\xi). \tag{33}$$

When calculating pairs $\left(U_0(\xi),\Omega_0(\xi)\right)$, $\left(U_1(\xi),\Omega_1(\xi)\right)$ in the case of functions $\bar{F}(\xi)$ that do not have discontinuities of the 1st kind, we use the standard procedures of the Runge–Kutta method of the 4th–5th order of accuracy. We determine the unknown constant p from the initial condition on the outer side surface of the cylinder: $\Omega_{clas}(1)=\Omega_0(1)+p\Omega_1(1)=0$.

If the functions $\bar{F}(\xi)$ have a discontinuity of the 1st kind on the surface $\xi=R_0$, we solve the Cauchy problems on the interval $[\bar{a},R_0]$, then we set the solutions found at the point $\xi=R_0$ as initial conditions for additional Cauchy problems, which we then solve on the half-interval $[R_0;1)$.

Having a solution to the problem of thermoelasticity in the classical formulation, we proceed to finding additional gradient terms for a small value of the parameter α based on the asymptotic WKB method [23]. To do this, we first compose an equilibrium equation in displacements by substituting into (20) the expressions for the total stresses (21), (22), which include the Cauchy stresses (28). Let us consider a homogeneous equation of equilibrium in displacements, which will represent a 4th order equation with variable coefficients and a small parameter at the highest derivative. According to [24], the WKB solution will be sought in the form:

$$U_{grad}(\xi)=D(\xi)e^{g(\xi)/\alpha}. \tag{34}$$

Let us expand the functions $D(\xi)$ and $g(\xi)$ by the parameter α:

$$\begin{aligned} D(\xi)&=D_0(\xi)+D_1(\xi)\alpha^1+D_2(\xi)\alpha^2+D_3(\xi)\alpha^3+\dots, \\ g(\xi)&=g_0(\xi)+g_1(\xi)\alpha^1+g_2(\xi)\alpha^2+g_3(\xi)\alpha^3+\dots. \end{aligned} \tag{35}$$

Let us substitute (34) into the equilibrium equation (20) taking into account expressions (35). Grouping the terms at the same degrees α, we obtain a sequence of problems, solving which we will find $D_0(\xi)=\dfrac{1}{\left(\bar{\lambda}(\xi)+2\bar{\mu}(\xi)\right)\sqrt{\xi}}$ and two values $g_0(\xi)$ in the form: $g_0(\xi)=\xi$ and $g_0(\xi)=-\xi$. Then the expressions for the gradient part of the displacements of the cylinder and the coating, due to the linearity of the problem, can be represented by the parameter as a linear combination of two WKB solutions:

$$U_{grad}^{(1)}(\xi)=\frac{1}{\left(\bar{\lambda}_1(\xi)+2\bar{\mu}_1(\xi)\right)\sqrt{\xi}}\left(B_1e^{-\xi/\alpha}+B_2e^{\xi/\alpha}\right), \qquad U_{grad}^{(2)}(\xi)=\frac{1}{\left(\bar{\lambda}_2(\xi)+2\bar{\mu}_2(\xi)\right)\sqrt{\xi}}\left(B_3e^{-\xi/\alpha}+B_4e^{\xi/\alpha}\right). \tag{36}$$

In formula (36), the constants B_1, B_2, B_3, B_4 are found by satisfying the boundary conditions (26), (27) for couple stresses and radial displacement gradients. Expressions $M_{rrr}=\alpha^2 S'_{rr}$ are made, where $S'_{rr}=\Omega'_{clas}+\Omega'_{grad}$. The gradient additive for radial stresses is determined by the formula: $\Omega_{grad}=\left(\bar{\lambda}+2\bar{\mu}\right)U'_{grad}+\bar{\lambda}\,U_{grad}/\xi$, and the expression (31) is used to find Ω'_{clas}. With now known constants, we can further find the radial displacements and radial Cauchy stresses, couple and total stresses.

5. Calculation results

Let us consider the results of calculating the distribution along the coordinate ξ of dimensionless temperatures, displacements, strains, Cauchy stresses, couple and total stresses in a cylinder with a heat-shielding coating. In all calculations it is accepted: $\overline{a}=0,6$; $R_0=0,9$; $\beta_0=0,4$; $\overline{\lambda}_1=\overline{\mu}_1=\overline{\gamma}_1=\overline{k}_1=1$.

On the example of a homogeneous coating, numerical verification of the proposed solution scheme was carried out. In this case, it was assumed in the calculations that the thermomechanical characteristics of the coating material are as follows: $\overline{\lambda}_2=2$; $\overline{\mu}_2=1{,}5$; $\overline{\gamma}_2=1$; $\overline{k}_2=0{,}1$. Analytical expressions for displacements in the case of a homogeneous coating, according to [15], were represented as:

$$U^{(1)}(\xi)=G_1\xi+\frac{G_2}{\xi}+\frac{\overline{\gamma}_1}{\left(\overline{\lambda}_1+2\overline{\mu}_1\right)\xi}\beta_0\int_{\overline{a}}^{\xi}\eta W^{(1)}(\eta)\,d\eta+G_3I_1\left(\frac{\xi}{\alpha}\right)+G_4K_1\left(\frac{\xi}{\alpha}\right),$$
$$U^{(2)}(\xi)=G_5\xi+\frac{G_6}{\xi}+\frac{\overline{\gamma}_2}{\left(\overline{\lambda}_2+2\overline{\mu}_2\right)\xi}\beta_0\int_{R_0}^{\xi}\eta W^{(2)}(\eta)\,d\eta+G_7I_1\left(\frac{\xi}{\alpha}\right)+G_8K_1\left(\frac{\xi}{\alpha}\right), \quad (37)$$

where $I_1(\xi/\alpha)$, $K_1(\xi/\alpha)$ are modified Bessel functions of the 1st and 2nd kind of the first order. In formula (37), the integration constants $G_1, ..., G_8$ were determined by satisfying the boundary conditions (25)–(27).

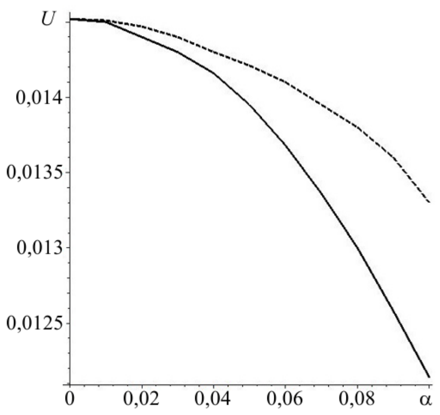

Fig. 1. Dependence of the radial displacement on the inner boundary of the cylinder on the gradient parameter α; exact (solid line) and approximate (dashed line).

Figure 1 shows graphs showing the dependence of the radial displacement on the inner boundary of the cylinder on the parameter $0\le\alpha\le0{,}1$. In this case, for comparison, the exact solution found by formulas (37) and the approximate solution using the shooting method and asymptotic formula (36) are shown. The maximum difference between the calculated displacement and the exact one was 10% at $\alpha=0{,}1$. For smaller values α, the solutions are closer to each other: for example, for $\alpha=0{,}01$ and they differ by 0.07%. It is found that the proposed numerical scheme for solving the problem for $\alpha<0{,}03$ gives an error in displacements and stresses that does not exceed 1%.

The following shows the distribution of the temperature (Fig. 2a) and displacement (Fig. 2b). For comparison, the solutions of the problem for displacement are given in the classical formulation for $\alpha=0$ and in the gradient formulation for $\alpha=0{,}03$.

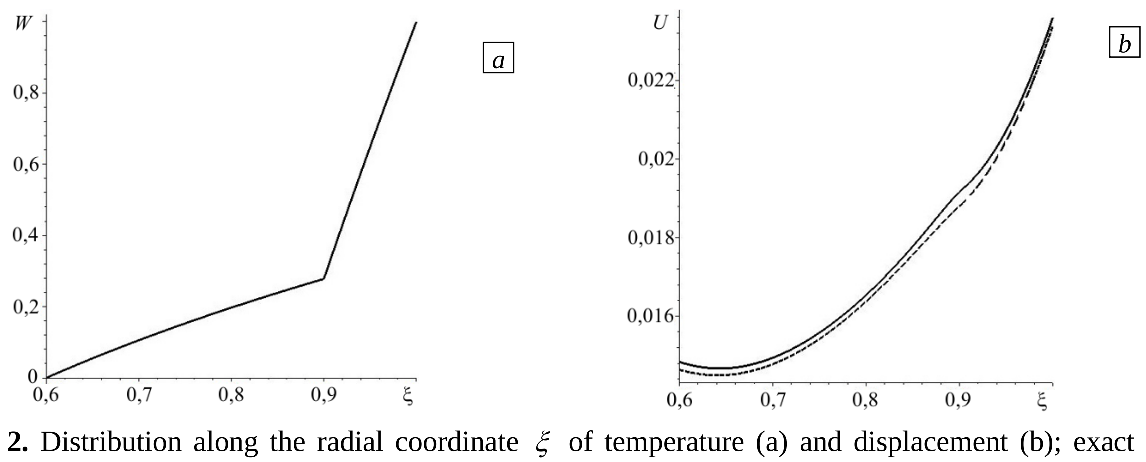

Fig. 2. Distribution along the radial coordinate ξ of temperature (a) and displacement (b); exact solution at $\alpha=0$ (solid line) and gradient solution at $\alpha=0{,}03$ (dashed line).

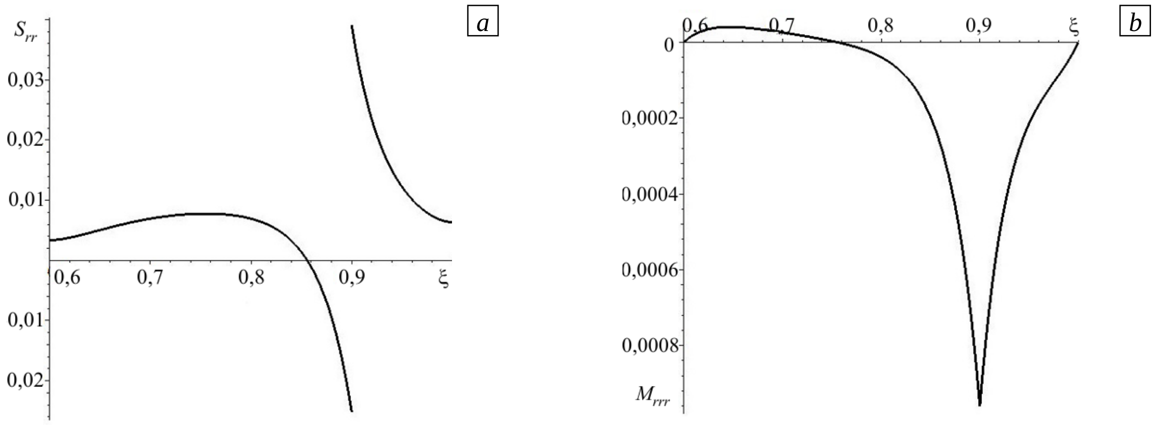

Fig. 3. Distribution along the radial coordinate ξ of the Cauchy radial stress (a) and couple stress M_{rrr} (b) at $\alpha = 0,03$.

Figure 3 contains the distribution of the Cauchy radial stress S_{rr} (Fig. 3a) and couple stress M_{rrr} (Fig. 3b) at the value of the scale parameter $\alpha = 0,03$. Figure 3a shows that the Cauchy radial stresses at the mating boundary suffer a discontinuity, which is due both to the difference in the thermomechanical characteristics of the cylinder and the coating, and with the continuity of displacements and their derivatives at the interface of dissimilar materials. It follows from Figure 3b that the couple stresses at low values M_{rrr} of the gradient parameter are ten times less than the Cauchy stresses. They peak at the mating surface.

Figure 4 shows the distribution of total stresses: radial (Fig. 4a) and circumferential (Fig. 4b), depending on the radial coordinate. In this case, the solid line shows the solution of the problem based on the classical formulation, the dashed line shows the solution of the problem in the gradient formulation at $\alpha = 0,03$. According to Figures 2 and 4, it can be concluded that with an increase in the scale parameter, the values of displacement and total circumferential stresses decrease, and the total radial stresses suffer a discontinuity.

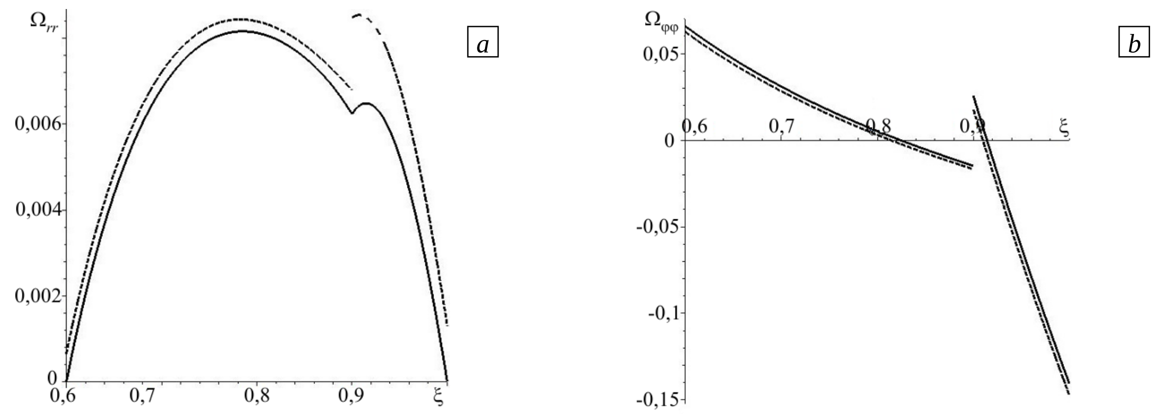

Fig. 4. Distribution along the radial coordinate ξ of total stresses: radial stress (a) and circumferential stress (b); exact solution at $\alpha = 0$ (solid line) and approximate solution at $\alpha = 0.03$ (dashed line).

In the second series of calculations, it was assumed that the cylinder coating is made of an inhomogeneous material with an inhomogeneity law that ensures a continuous change in thermomechanical characteristics when passing through the mating surface in the form: $\bar{\lambda}_2(\xi) = 1 + (10\xi - 9)^N$, $\bar{\mu}_2(\xi) = 1 + 0,5(10\xi - 9)^N$, $\bar{\gamma}_2(\xi) = 1$, $\bar{k}_2(\xi) = 1 - 0,9(10\xi - 9)^N$, $N = 1, 2, \ldots$.

Let us show the results of calculation of displacements, deformations and stresses with the inhomogeneity index $N = 1$. Figure 5 shows the radial distributions of displacements (Fig. 5a) and total circumferential stress (Fig. 5b). It follows from the figure that, in the case of an inhomogeneous coating, in contrast to the case of a homogeneous coating, the total circumferential stresses do not undergo discontinuity, but change continuously, while an increase in the gradient parameter, just as in the case of a uniform coating, leads to a decrease in displacements and circumferential stresses.

Figure 6 shows the radial distribution of deformations $dU/d\xi$ in the "cylinder-coating" system for a homogeneous and inhomogeneous ($N=1$) coating. It can be seen from the figure that at $\alpha=0$, in the case of a homogeneous coating, the deformations are discontinuous, while in the case of an inhomogeneous coating, they are continuous. In the gradient formulation, the deformations are continuous for both types of coverage.

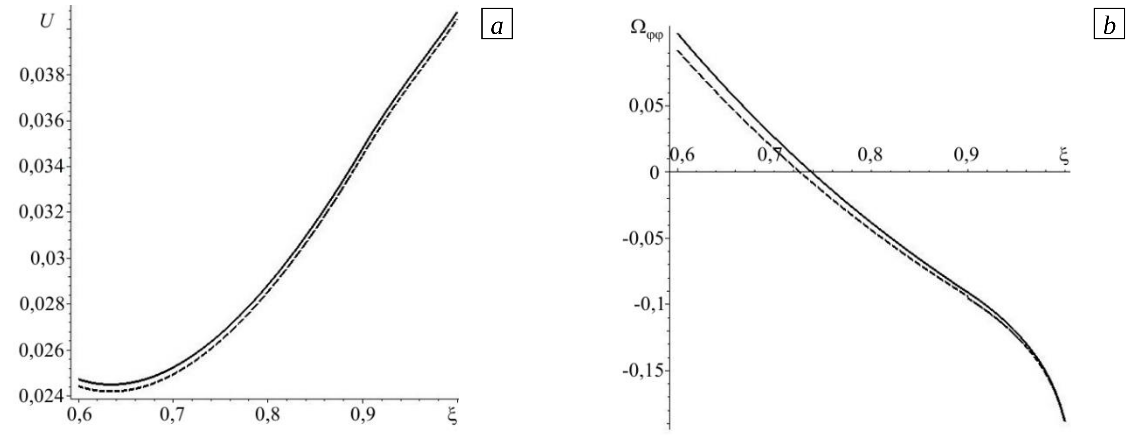

Fig. 5. Distribution along the radial coordinate ξ of displacement (a) and total circumferential stress (b); exact solution at $\alpha=0$ (solid line) and approximate solution at $\alpha=0,03$ (dashed line).

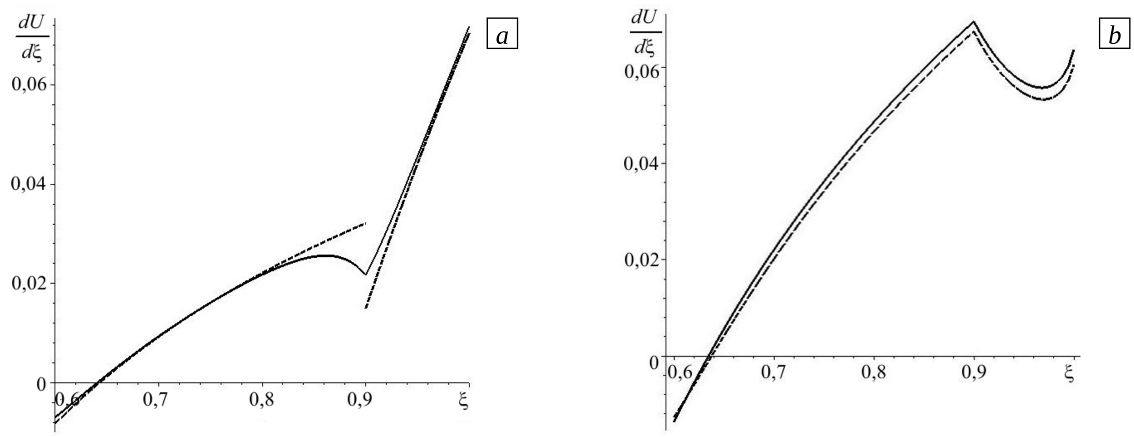

Fig. 6. Distribution along the radial coordinate ξ of deformations $dU/d\xi$ in the cases of homogeneous (a) and inhomogeneous (b) coating; exact solution at $\alpha=0$ (solid line) and approximate solution at $\alpha=0,03$ (dashed line).

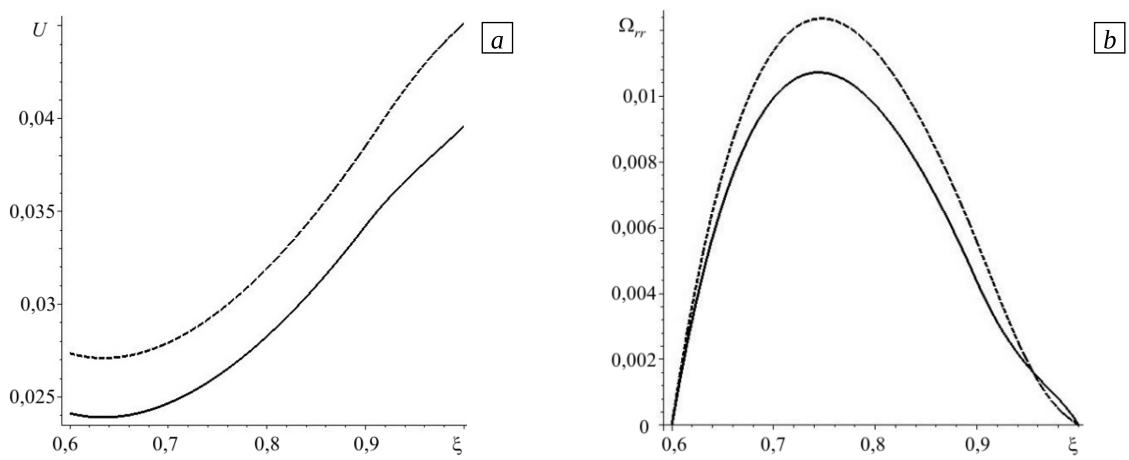

Fig. 7. Distribution along the radial coordinate ξ of the displacement (a) and the total radial stress (b) at $\alpha=0,03$ and different values of the inhomogeneity parameter N: 1 (solid line) and 2 (dashed line).

In addition, the influence of various heterogeneity laws, characterized by the parameter N, on the distribution of displacements and stresses was studied. Figure 7 shows graphs of the radial distribution of displacements and total radial stresses at $\alpha = 0,03$ and different values of the parameter N. It follows from the figure that the law of inhomogeneity has a strong influence on the distribution of displacements and stresses.

6. Conclusions

The problem of gradient thermoelasticity for a cylinder with a thermal protective coating is studied. Displacements and stresses are presented as the sum of solutions of the classical thermoelastic problem and gradient terms. The problem of thermoelasticity in the classical formulation is solved by the shooting method. The gradient terms are obtained using the asymptotic WKB method. The difference from each other in the distributions of displacements and stresses along the radial coordinate, calculated according to the classical theory and found using the gradient theory of thermoelasticity, is shown. It was found that an increase in the value of the gradient parameter reduces the values of radial displacements and total stresses. The couple stresses at small values of the gradient parameter are much less than the total stresses. The Cauchy stress jump in the vicinity of the interface of dissimilar materials is explained by the continuity of displacements and their gradients. The influence of the inhomogeneity parameter in the power law, which models the thermomechanical characteristics of an inhomogeneous coating, on the distributions of displacements and total stresses along the radial coordinate is studied.

Taking into consideration the influence of the gradient parameter in the analysis of the stress-strain state of a coated hollow cylinder is of great practical importance in calculating the strength and assessing the loss of stability of a thermal protective coating. The developed approach can be applied to find an approximate analytical solution to the problem of gradient thermoelasticity for a finite cylinder.

References

1. Padture N.R., Gell M., Jordan E.H. Thermal barrier coatings for gas-turbine engine applications. *Science*, 2002, vol. 296, pp. 280-284. https://doi.org/10.1126/science.1068609
2. Bialas M. Finite element analysis of stress distribution in thermal barrier coatings. *Surf. Coating Tech.*, 2008, vol. 202, pp. 6002-6010. https://doi.org/10.1016/j.surfcoat.2008.06.178
3. Vatulyan A., Nesterov S., Nedin R. Regarding some thermoelastic models of «coating-substrate» system deformation. *Continuum Mech. Thermodyn.*, 2020, vol. 32, pp. 1173-1186. https://doi.org/10.1007/s00161-019-00824-9
4. Toupin R.A. Elastic materials with couple-stresses. *Arch. Rational Mech. Anal.*, 1962, vol. 11, pp. 385-414. https://doi.org/10.1007/BF00253945
5. Mindlin R.D. Micro-structure in linear elasticity. *Arch. Rational Mech. Anal.*, 1964, vol. 16, pp. 51-78. https://doi.org/10.1007/BF00248490
6. Ahmadi G., Firoozbakhsh K. First strain gradient theory of thermoelasticity. *Int. J. Solid Struct.*, 1975, vol. 11, pp. 339-345. https://doi.org/10.1016/0020-7683(75)90073-6
7. Lur'ye M.V. Zadachi Lame v gradiyentnoy teorii uprugosti [Lame problems in the gradient theory of elasticity]. *DAN SSSR – Dokl. Akad. Nauk SSSR*, 1968, vol. 181, no. 5, pp. 1087-1089.
8. Altan B.S., Aifantis E.C. On some aspects in the special theory of gradient elasticity. *JMBM*, 1997, vol. 8, pp. 231-282. https://doi.org/10.1515/JMBM.1997.8.3.231
9. Askes H., Aifantis E.C. Numerical modeling of size effects with gradient elasticity – Formulation, meshless discretization and examples. *Int. J. Fruct.*, 2002, vol. 117, pp. 347-358. https://doi.org/10.1023/A:1022225526483
10. Askes H., Aifantis E.C. Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. *Int. J. Solid. Struct.* 2011. Vol. 48. P. 1962-1990. https://doi.org/10.1016/j.ijsolstr.2011.03.006
11. Aifantis E.C. Gradient effects at the macro, micro and nano scales. *JMBM*, 1994, vol. 5, pp. 335-353. https://doi.org/10.1515/JMBM.1994.5.3.355
12. Lur'ye S.A., Belov P.A., Rabinskiy L.N., Zhavoronok S.I. *Masshtabnyye effekty v mekhanike sploshnykh sred. Materialy s mikro- i nanostrukturoy* [Scale effects in continuum mechanics. Materials from micro- and nanostructures]. Moscow, Izd-vo MAI, 2011. 160 p.
13. Gao X.-L., Park S.K. Variational formulation of a simplified strain gradient elasticity theory and its application to a pressurized thick-walled cylinder problem. *Int. J. Solid. Struct.*, 2007, vol. 44, pp. 7486-7499. https://doi.org/10.1016/j.ijsolstr.2007.04.022

14. Chu L., Dui G. Exact solutions for functionally graded micro-cylinders in first gradient elasticity. *Int. J. Mech. Sci.*, 2018, vol. 148, pp. 366-373. https://doi.org/10.1016/j.ijmecsci.2018.09.011
15. Sadeghi H., Baghani M., Naghdabadi R. Strain gradient thermoelasticity of functionally graded cylinders. *Scientia Iranica B*, 2014, vol. 21, pp. 1415-1423.
16. Hosseini M., Dini A., Eftekhari M. Strain gradient effects on the thermoelastic analysis of a functionally graded micro-rotating cylinder using generalized differential quadrature method. *Acta Mech.*, 2017, vol. 228, pp. 1563-1580. https://doi.org/10.1007/s00707-016-1780-5
17. Lurie S.A., Solyaev Yu.O., Rabinsky L.N., Kondratova Yu.N., Volov M.I. Simulation of the stress-strain state of thin composite coating based on solutions of the plane problem of strain-gradient elasticity for layer. *Vestnik PNIPU. Mekhanika – PNRPU Mechanics Bulletin*, 2013, no. 1, pp. 161-181.
18. Li A., Zhou S., Zhou S., Wang B. A size-dependent bilayered microbeam model based on strain gradient elasticity theory. *Compos. Struct.*, 2014, vol. 108, pp. 259-266. https://doi.org/10.1016/j.compstruct.2013.09.020
19. Li A., Zhou S., Zhou S., Wang B. A size-dependent model for bi-layered Kirchhoff micro-plate based on strain gradient elasticity theory. *Compos. Struct.*, 2014, vol. 113, pp. 272-280. https://doi.org/10.1016/j.compstruct.2014.03.028
20. Fu G., Zhou S., Qi L. The size-dependent static bending of a partially covered laminated microbeam. *Int. J. Mech. Sci.*, 2019, vol. 152, pp. 411-419. https://doi.org/10.1016/j.ijmecsci.2018.12.037
21. Lurie S.A., Pham T., Soliaev J.O. Gradient model of thermoelasticity and its application for the modeling of thin layered composite structures. *MKMK – Journal of Composite Mechanics and Design*, 2012, vol. 18, no. 3, pp. 440-449.
22. Vatulyan A.O., Nesterov S.A. On the deformation of a composite rod in the framework of gradient thermoelasticity. *Materials Physics Mechanics*, 2020, vol. 46, pp. 27-41. https://doi.org/10.18149/MPM.4612020_3
23. Stoer J., Bulirsch R. *Introduction to numerical analysis*. Springer, 2002. 746 p. https://doi.org/10.1007/978-0-387-21738-3
24. Maslov V.P. *Kompleksnyy metod VKB v nelineynykh uravneniyakh* [Complex WKB method in nonlinear equation]. Moscow, Nauka, 1977. 384 p.
25. Vatulyan A.O., Nesterov S.A. On the identification problem of the thermomechanical characteristics of the finite functionally graded cylinder. *Izv. Sarat. Un-ta. Nov. ser. Ser. Matematika. Mekhanika. Informatika – Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2021, vol. 21, no. 1, pp. 35-47. https://doi.org/10.18500/1816-9791-2021-21-1-35-47

The authors declare no conflict of interests.
The paper was received on 14.04.2021.
The paper was accepted for publication on 22.06.2021.