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PLANE VORTEX FLOW IN A CYLINDRICAL LAYER STS

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This paper presents brief analysis of publications dealing with the issues of experimental and theoretical studies of spiral fluid flows. The flows of this kind occur, in particular, in the vicinity of drain holes and are also observed in nature as storms and tornadoes. A mathematical modeling of a plane flow in a cylindrical layer has been carried out assuming that the viscous incompressible fluid is supplied along the normal to its outer surface and, accordingly, the vortex flow happens through its inner surface. The well-known general solution of the problem of vortex flow in unlimited space has been taken as a basis. A variant setting of the two boundary conditions for determining the velocity azimuthal component is proposed. The first boundary condition is the requirement for the azimuthal velocity component absence at the cylindrical layer entrance. The second, less obvious, boundary condition is adopted in the form when the strain rate tensor second invariant value is set at the cylindrical layer entrance. The rationale for such variant setting of the second boundary condition is given. Finally, it became possible to find an accurate solution to the problem in a general setting. It is shown that at the layer exit, the velocity azimuthal component, which was initially absent at the layer entrance, can exceed the radial velocity component by an order of magnitude or more. A family of streamlines is constructed for the examined flow in the polar coordinate system. It is shown that, in the general case, the streamlines for the flow under consideration must have an inflection point. The numerical dependence of the inflection point position radial coordinate on the Reynolds number was obtained. It is also shown that in the cylindrical layer exit vicinity the fluid pressure takes on lower values than for the purely radial flow case. An alternative formulation of the second boundary condition for the pressure is presented for determining the integration constants in the azimuthal velocity component expression.

***Key words**:* viscous incompressible fluid, plane spiral flow, boundary conditions, the second invariant of the strain rate tensor, streamlines, inflection point

1. Introduction

Spiral flows are of interest from the point of view of modeling natural phenomena and the functioning of technical devices. Such flows occur, for example, in drain holes. Various issues related to the experimental study of the flow characteristics and the conditions for the formation of funnels during the outflow of liquids are discussed in [1–6]. Natural spiral flows in the atmosphere and water bodies of the Earth, which are realized in the form of funnels, sandstorms and tornadoes, are well known [7–10]. A detailed review of such studies is given in [11].

Theoretical study, as well as mathematical modeling of the dynamics of vortex flows are reflected in extensive bibliography. The theory of the dynamics of a rotating fluid and its aspects as applied to some technical problems are presented in monographs [12–15]. An analysis of various options for setting the boundary conditions is presented in [16]. A number of important results were obtained in [17–30]. In many cases, vortex flows are considered in areas bounded by cylindrical surfaces.

In [31], the problem of gas flow through a cylindrical surface in a rotating coordinate system is considered. It is shown that after the onset of a radial flow with gas flow into the cylinder, the formation of a spiral flow immediately begins under the action of the Coriolis force. The solution of the problem is carried out within the framework of the model of a compressible ideal gas for the case when neglect of its viscosity is permissible.

The problem of the rotational Taylor – Couette flow for a viscous fluid in the gap between coaxial cylinders is well known. Since Taylor's results [32], which were further developed in the works of various authors [33–36], it has been found that when the angular velocity of the inner cylinder rotation exceeds a certain critical value, the liquid particles "generate" components transverse to the previously existing circular streamlines velocity, and vortex structures begin to form in the gap. In this case, the streamlines of such structures are spirals wound on tori.

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The problem of flow between two coaxial semi-infinite cylinders with a gap between the cylinders plugged at one end was considered in [37]. An asymptotic flow is simulated, which can be formed at a sufficiently large distance from the free end section of coaxial cylinders. It is shown that, in the axisymmetric case, the structure of closed streamlines in the gap in the direction of the longitudinal axis of the cylinders turns cellular. When the gap width is negligible in comparison with the radius of the outer cylinder, the asymptotic solution is obtained analytically.

In [38, 39], one class of solutions of the equations of the dynamics of a viscous fluid in a cylindrical coordinate system for a steady flow is considered, provided that the radial and azimuthal components of the velocity depend only on the radial coordinate.

This class also includes the problem of a vortex axisymmetric flow of a viscous fluid in an unlimited space; its general solution for the radial and azimuthal velocity components is presented in [40]. The resulting general solution for the azimuthal velocity component contains two undefined integration constants. In [40], a particular case of this general solution is given, where one of the integration constants is a priori taken to be zero, and the second constant is determined by means of a given flow circulation.

It should be noted that the general solution from [40] can be taken as a basis when considering the problem of a plane flow in a cylindrical layer when fluid is supplied normally (radially) through the outer cylindrical surface and, accordingly, with a vortex flow through the inner surface. However, when obtaining the final expression for the distribution of the azimuthal velocity component, it becomes necessary to specify two boundary conditions for determining the two integration constants. In this formulation of the problem, the form of at least one of the two necessary boundary conditions for the azimuthal component of the velocity is not entirely obvious.

In this paper, for a vortex axisymmetric flow of a viscous fluid, we propose options for writing two boundary conditions for the azimuthal component of the velocity, which makes it possible to obtain an exact solution to the model problem of vortex flow, describing the distributions of the velocity and pressure components in a cylindrical layer when the fluid is normally supplied through the external surface. In addition, the analysis of the obtained solution is carried out and the features of the velocity field and pressure distribution are discussed.

2. Setting of the problem

Consider a model problem of a plane steady axisymmetric laminar flow of a viscous incompressible fluid in a cylindrical layer. Suppose that the liquid is supplied normally through the layer outer surface, which has a radius of R_2. There is a constant pressure of P_2 maintained on it. In this setting, the drainage of the liquid occurs through the inner surface of the radius R_1.

Let us introduce a cylindrical coordinate system in the traditional way. Taking into account the following assumptions: $u_z \equiv 0$, $\partial/\partial\varphi \equiv 0$, $\partial/\partial z \equiv 0$, we write down the equations of dynamics and the condition of continuity of the flow in dimensionless form [41]:

$$. u'_r \cdot \frac{\partial u'_r}{\partial r'} - \frac{u'^2_\varphi}{r'} = -\frac{1}{2}\cdot\frac{\partial P'}{\partial r'} + \frac{1}{\mathrm{Re}}\cdot\left(\frac{\partial^2 u'_r}{\partial r'^2} + \frac{1}{r'}\cdot\frac{\partial u'_r}{\partial r'} - \frac{u'_r}{r'^2}\right)., \tag{1}$$

$$u'_r \cdot \frac{\partial u'_\varphi}{\partial r'} + \frac{u'_r \cdot u'_\varphi}{r'} = \frac{1}{\mathrm{Re}}\cdot\left(\frac{\partial^2 u'_\varphi}{\partial r'^2} + \frac{1}{r'}\cdot\frac{\partial u'_\varphi}{\partial r'} - \frac{u'_\varphi}{r'^2}\right), \tag{2}$$

$$\frac{1}{r'}\cdot\frac{\partial\left(r'\cdot u'_r\right)}{\partial r'} = 0\,. \tag{3}$$

Here: $r' = \frac{r}{R_1}$, $u'_r\left(r'\right) = \frac{u_r}{V_1}$, $u'_\varphi\left(r'\right) = \frac{u_\varphi}{V_1}$, $P' = \frac{2P}{\rho V_1^2}$ — dimensionless values (further on noted by a top stroke); $\mathrm{Re} = \frac{\rho V_1 R_1}{\mu} = \frac{\rho V_2 R_2}{\mu} = \frac{\rho Q}{2\pi\mu}$ — Reynolds number; r — radial coordinate; ρ, μ — density and dynamic viscosity of the fluid; u_r, u_φ — radial and azimuthal components of flow velocity; P — pressure; V_1, V_2 — known values of the modulus of the radial velocity component at the inner and outer cylindrical boundaries of the layer; Q — absolute volumetric flow rate of fluid per unit length along the Oz axis of cylindrical system of coordinates.

For the solution of the system of equations (1)–(3), it is necessary to require the fulfillment of four boundary conditions. The first two boundary conditions for the radial component of velocity and pressure have the following form:

$$r' = R_2' : \qquad u_r' = -V_2' , \qquad P' = P_2' . \tag{4}$$

Note that (4) act as boundary conditions for the considered problem, but only in the trivial case of a merely radial flow ($u_\varphi \equiv 0$). As for two more boundary conditions necessary to determine the azimuthal velocity component, they will be proposed below.

A distinctive feature of the system (1)–(3) is that it allows the definition of the required functions u_r', u_φ', P' based on the sequential solution of its equations in reverse order: (3), (2), (1). In this setting, at the first step the solution of the equation (3) for the radial component of velocity considering the boundary condition (4), as well as the correlation $V_1 \cdot R_1 = V_2 \cdot R_2$ evident for the problem under consideration, is always the following [42]:

$$u_r' = -\frac{1}{r'} . \tag{5}$$

At the next step, having substituted (5) for (2), we come to a differential equation relative to the azimuthal component of the velocity, obtained earlier in [40] and written in the dimensionless notation adopted above:

$$\frac{d^2 u_\varphi'}{dr'^2} + (\mathrm{Re}+1) \cdot \frac{1}{r'} \cdot \frac{du_\varphi'}{dr'} + (\mathrm{Re}-1) \cdot \frac{u_\varphi'}{r'^2} = 0 . \tag{6}$$

General solution (6) is obtained in [40], and in a dimensionless form may be manifested as follows:

$$u_\varphi'(r) = \frac{C_1}{r'} + \frac{C_2}{r'^{\mathrm{Re}-1}} , \tag{7}$$

where C_1, C_2 are undefined integration constants.

Note that the authors [40] within the framework of general solution (7) limited themselves to considering only a particular case, when took $C_2 = 0$. In this case, the first constant of integration was determined through the specified circulation K and, considering dimensionless form of notation (7), equaled $C_1 = K / (2\pi R_1 V_1)$. A characteristic feature of such a particular case is that in the flow region the azimuthal component of the velocity cannot be exactly zero.

With finite dimensions of the flow region in the cylindrical layer $R_1 \le r \le R_2$, determination of the final form of the solution of the problem under consideration based on the general solution (7) with $C_2 \ne 0$ involves setting two boundary conditions for the azimuthal velocity component. Taking into account the radial supply of fluid into the cylindrical layer through its outer surface, the first boundary condition can be obtained by requiring the absence of such a velocity component on the outer cylindrical surface:

$$r' = R_2' : \qquad u_\varphi' = 0 . \tag{8}$$

The second boundary condition for the azimuthal velocity component at the outer boundary of the cylindrical layer is less obvious.

Formally, determination of the azimuthal velocity component from a second-order differential equation (6) within the framework of the Cauchy problem statement requires satisfying one more boundary condition for the first derivative du'_φ / dr'. Note that the first derivative of the azimuthal velocity along the radial coordinate is directly related to the second invariant I_2 of the strain rate tensor. Then, at the model level, the above-mentioned formalism can be realized by setting on the outer boundary the value I_{2b} of the strain rate tensor second invariant modulus:

$$r' = R'_2: \qquad |I'_2| = I'_{2b}. \tag{9}$$

In this setting, $I'_2 = \dfrac{I_2}{I_{2rad}}$, $I'_{2b} = \dfrac{I_{2b}}{I_{2rad}}$, $I_{2rad} = \left(\dfrac{V_2}{R_2}\right)^2$, $I_2(r) = \varepsilon_{rr} \cdot \varepsilon_{\varphi\varphi} - \varepsilon_{r\varphi}^2$, $\varepsilon_{rr} = \dfrac{du_r(r)}{dr}$, $\varepsilon_{\varphi\varphi} = \dfrac{u_r(r)}{r}$, $\varepsilon_{r\varphi} = \dfrac{r}{2} \cdot \dfrac{d}{dr}\left(\dfrac{u_\varphi(r)}{r}\right)$, where I_{2b} is the modulus of the second invariant of the strain rate tensor on the outer boundary of the cylindrical layer, defined considering (5) for the case of a purely radial flow ($u_\varphi \equiv 0$), ε_{rr}, $\varepsilon_{\varphi\varphi}$, $\varepsilon_{r\varphi}$ are strain rate tensor components.

Taking into account (5) and (8), boundary condition (9) – which is the second condition for the azimuthal component of the velocity, – after some transformations, can be represented as follows:

$$r' = R'_2: \qquad \frac{du'_\varphi}{dr'} = \pm \frac{2}{R'^2_2} \cdot \sqrt{I'_{2b} - 1}. \tag{10}$$

The options for choosing the sign incorporated in (10) indicate the possibility of realizing two equivalent and opposite directions of "twisting" of the flow within the framework of solving the same initial system of equations. The concretization of the flow direction in the course of its realization, apparently, can be determined by the features of the initial stage of its "launch" or by additional factors.

It is important that boundary condition (10) makes sense only for values $I'_{2b} \geq 1$. The physical interpretation of the formulation of the boundary condition for the azimuthal velocity component in a similar form in the sense of its connection with more traditional mechanical characteristics will be given below.

Thus, the solution of the considered problem of a plane flow of an incompressible fluid (of a vortex flow) when the fluid is supplied into a cylindrical layer along the normal to its outer surface must satisfy the boundary conditions (4), (8) и (10).

3. Distributions of velocity and pressure components

The solution to the problem for the radial velocity component has the well-known form given above (5). Now, taking into account the boundary conditions (8), (10), let us find the integration constants in (7) and, after the appropriate transformations, obtain an expression for determining the azimuthal component of the velocity:

$$u'_\varphi(r') = \pm \frac{2\sqrt{I'_{2b} - 1}}{\mathrm{Re} - 2} \cdot \frac{1}{r'} \cdot \left[1 - \left(\frac{R'_2}{r'}\right)^{\mathrm{Re}-2}\right]. \tag{11}$$

Note that the case when the second invariant of the strain rate tensor at the outer boundary of the cylindrical layer takes on a value corresponding to a purely radial flow ($I'_{2b}=1$), immediately leads from (11) to the expected result: $u'_\varphi(r')\equiv 0$.

Now, substituting (5) and (11) into equation (1), after integration, taking into account the boundary condition (4), we find the expression for the pressure distribution:

$$P'(r') = P'_2 - f_1(r') + \frac{4(I'_{2b}-1)}{(\mathrm{Re}-2)^2}\cdot\left\{-f_1(r') + \frac{4R_2'^{\mathrm{Re}-2}}{\mathrm{Re}}\cdot f_2(r') - \frac{R_2'^{2\mathrm{Re}-4}}{(\mathrm{Re}-1)}\cdot f_3(r')\right\}. \quad (12)$$

Here, for the sake of brevity, we adopt the following notation:

$$f_1(r') = \left(\frac{1}{r'^2} - \frac{1}{R_2'^2}\right), \qquad f_2(r') = \left(\frac{1}{r'^{\mathrm{Re}}} - \frac{1}{R_2'^{\mathrm{Re}}}\right), \qquad f_3(r') = \left(\frac{1}{r'^{2\mathrm{Re}-2}} - \frac{1}{R_2'^{2\mathrm{Re}-2}}\right).$$

Assuming a purely radial ($u_\varphi \equiv 0$) steady flow in a cylindrical layer $R_1 \le r \le R_2$, when $I'_{2b}=1$, we obtain from (12) the following result for the pressure distribution:

$$P'_{rad}(r') = P'_2 - f_1(r'). \quad (13)$$

For specific values of the Reynolds number Re, which is a parameter of the differential equation (6), the form of the general solution (7) changes. In this case, expressions (11) and (12) for the distributions of the azimuthal component of velocity and pressure also become different. Let us analyze the solution of the vortex flow problem for particular values of Re.

Suppose $\mathrm{Re}=1$. In such setting equation (6) is simplified, and its general solution acquires the following form:

$$u_\varphi'^{(1)}(r) = \frac{C_1^{(1)}}{r'} + C_2^{(1)}.$$

Having found the constants of integration $C_1^{(1)}$, $C_2^{(1)}$, and considering boundary conditions (8), (10), we arrive at the following expression for the azimuthal component of the velocity:

$$u_\varphi'^{(1)}(r') = \mp\frac{2\sqrt{I'_{2b}-1}}{r'}\cdot\left[1-\left(\frac{r'}{R'_2}\right)\right]. \quad (14)$$

The same result can be obtained by direct substitution $\mathrm{Re}=1$ into the solution (11).

As for the pressure distribution, for this particular value of Re, after substitution (5), (14) into (1) and solution of the latter with consideration to the boundary condition (4), it becomes determined as follows:

$$P'^{(1)}(r') = P'_2 - f_1(r') + 4(I'_{2b}-1)\cdot\left\{-f_1(r') + \frac{4}{R'_2}\cdot\left(\frac{1}{r'}-\frac{1}{R'_2}\right) + \frac{2}{R_2'^2}\cdot\ln\left(\frac{r'}{R'_2}\right)\right\}. \quad (15)$$

If in (12) we pass to the limit with $\mathrm{Re}\to 1$ and disclose the resulting uncertainty according to L'Hôpital's rule, we come precisely to the expression (15):

$$\lim_{\mathrm{Re}\to 1}\left\{P'(r')\right\} = P'^{(1)}(r').$$

Analogically, let us consider another particular case, when $\mathrm{Re}=2$. Substituting this value in (6), we obtain a slightly different form of the differential equation, the general solution of which for the azimuthal component of the velocity also looks different:

$$u_\varphi'^{(2)} = \frac{C_1^{(2)}}{r'} + \frac{C_2^{(2)}\cdot\ln(r')}{r'},$$

where $C_1^{(2)}$, $C_2^{(2)}$ are integration constants, corresponding to $\mathrm{Re}=2$.

Satisfying the boundary conditions (8), (10), we obtain the following expression for the distribution of the azimuthal velocity component:

$$u_\varphi'^{(2)} = \pm\frac{2\sqrt{I_{2b}'-1}}{r'}\cdot\ln\left(\frac{r'}{R_2'}\right). \tag{16}$$

Substituting (5) and (16) into (1) and considering boundary condition (4), we obtain pressure distribution for the concerned number Re:

$$P'^{(2)}(r') = P_2' - \left(1+2\left(I_{2b}'-1\right)\right)\cdot f_1(r') - \frac{4\left(I_{2b}'-1\right)}{r'^2}\cdot\left[1+\ln\left(\frac{r'}{R_2'}\right)\right]\cdot\ln\left(\frac{r'}{R_2'}\right). \tag{17}$$

The passage in (11) and (12) to the limit with $\mathrm{Re}\to 2$ and the disclosure of the resulting uncertainties lead precisely to the just found expressions (16) и (17):

$$\lim_{\mathrm{Re}\to 2}\left\{u_\varphi'(r')\right\} = u_\varphi'^{(2)}(r'), \qquad \lim_{\mathrm{Re}\to 2}\left\{P'(r')\right\} = P'^{(2)}(r').$$

In other words, the solutions corresponding to the values $\mathrm{Re}=1$ and $\mathrm{Re}=2$, are contained in solutions (11) and (12). This is due to the fact that the form (6) for the vortex flow problem satisfies the theorem on the continuous dependence of the solution of the differential equation on the parameter; here its function is performed by the Reynolds number.

4. Analysis and some features of a steady plane spiral flow

First, let us note the following: having analyzed relations (5) and (11), we can see that as the point considered in the flow region approaches the inner boundary of the cylindrical layer, the azimuthal component of the velocity increases significantly. As an example, Figure 1 shows the nature of the change in the modulus of the ratio of the azimuthal and radial velocity components depending on the radial coordinate for different values of the Reynolds number:

$$\gamma_u'(r') = \left|\frac{u_\varphi'(r')}{u_r'(r')}\right| = \frac{2\sqrt{I_{2b}'-1}}{\mathrm{Re}-2}\cdot\left[\left(\frac{R_2'}{r'}\right)^{\mathrm{Re}-2}-1\right]. \tag{18}$$

The influence of the parameter R_2' on the value of relation (18) depending on the Reynolds number at the exit from the cylindrical layer with $r'=1$ is shown in the graphs in Figure 2.

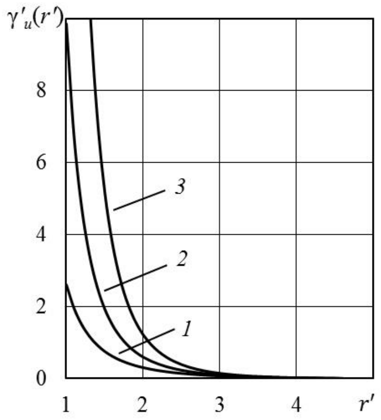

Fig. 1. Change in the ratio between the azimuthal and radial velocity components depending on the radial coordinate with $I'_{2b}=1{,}001$; $R'_2=5$ and different values of Re: 5 (curve *1*); 6 (*2*); 7 (*3*).

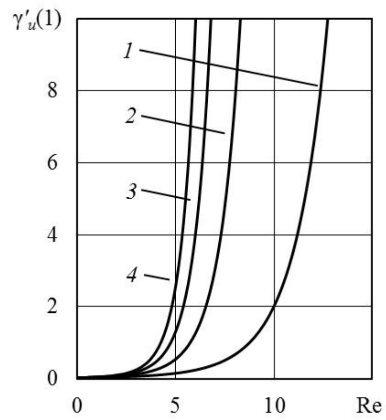

Fig. 2. Change of the ratio between the azimuthal and radial velocity components at the exit from the cylindrical layer depending on Re with $I'_{2b}=1{,}001$ and different values of R'_2: 2 (curve *1*); 3 (*2*); 4 (*3*); 5 (*4*).

The curves in Figures 1 and 2 show that the azimuthal component of the velocity in the immediate vicinity of the exit from the cylindrical layer begins to dominate over the radial component. In this case, for the studied sets of numerical values of the initial parameters, the azimuthal component of the velocity at the exit from the cylindrical layer can exceed the radial component by an order of magnitude or more.

Note that precisely this type of correspondence between the velocity components, at least at the qualitative level, is characteristic of real natural vortex currents in the atmosphere, such as sandstorms and tornadoes. For comparison, let us point that solution (7) known from [40] with $C_2=0$ predetermines everywhere in the flow area one and the same value of ratio between azimuthal and radial velocity components, that is: $\gamma'_u(r')=\left|\dfrac{u'_\varphi(r')}{u'_r(r')}\right|=\dfrac{K}{Q}\equiv \mathrm{const}$.

Let us consider some features of the streamline family for the obtained distribution of the velocity vector components (5), (11) in the vortex flow problem in the case of a steady plane spiral flow. In a polar coordinate system, the equation for determining streamlines can be written as follows:

$$\frac{d\varphi}{dr'}=\frac{1}{r'}\cdot\frac{u'_\varphi(r')}{u'_r(r')}\,. \tag{19}$$

The boundary condition for this equation can be set as

$$r'=R'_2:\qquad \varphi=\varphi_0\,, \tag{20}$$

where φ_0 is an angle, determining the location of the initial point on the outer boundary of the cylindrical layer, which a streamline is to be drawn through.

Considering (5) and (11), taking into account the boundary condition (20), we come to the solution of equation (19) of the form:

$$\varphi(r')=\varphi_0\pm\frac{2\sqrt{I'_{2b}-1}}{(\mathrm{Re}-2)^2}\cdot\left\{\left[\left(\frac{R'_2}{r'}\right)^{\mathrm{Re}-2}-1\right]-\ln\left(\frac{R'_2}{r'}\right)^{\mathrm{Re}-2}\right\}. \tag{21}$$

A distinctive feature of the velocity field of the considered steady flow is that the streamlines satisfying solution (21) have the shape of spirals, which allow the presence of an inflection point. In passing, we note that most of the known plane spiral curves (spirals of Archimedes, Fibonacci, logarithmic, hyperbolic, and others) do not have an inflection point. Among the few analogues, perhaps only the so-called spiral of the Roman rod (spiral Lituus) can be distinguished, each of the two branches of which also has an inflection point.

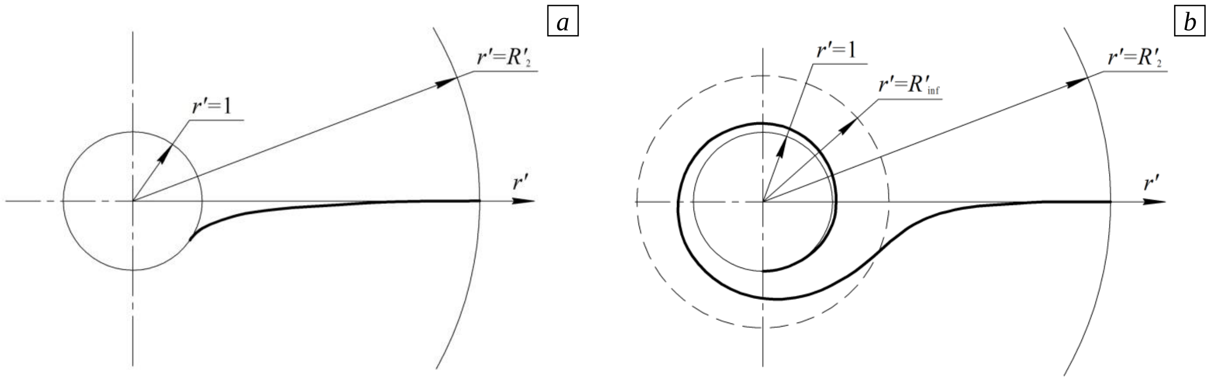

Fig. 3. Streamline for a steady flow in a cylindrical layer with $I'_{2b}=1{,}001$, $R'_2=5$, $\varphi_0=0$ for $\mathrm{Re}=4{,}25$, $R'_{\inf}<1$ (*a*) and for $\mathrm{Re}=7$, $R'_{\inf}=1{,}813$ (*b*).

Typical examples of two different versions of streamlines constructed in accordance with (21) for relatively small and sufficiently large values of the Reynolds number are shown in Figure 3. The main difference between the two types of streamlines presented here is as follows. For relatively small values of the Reynolds number, the streamline does not have an inflection point in the range of variation of the radial coordinate $1\le r'\le R'_2$ (Fig. 3*a*). In the case of sufficiently large values of the Reynolds number, the streamline takes on such a shape that an inflection point appears at a certain value of the radial coordinate $r'=R'_{\inf}\in\left[1;\ R'_2\right]$ (Fig. 3*b*).

The condition for the existence of an inflection point on an arbitrary line in a polar coordinate system is defined by the following differential relation:

$$r'\cdot\frac{d^2\varphi}{dr'^2}+\left[2+\left(r'\cdot\frac{d\varphi}{dr'}\right)^2\right]\cdot\frac{d\varphi}{dr'}=0\,. \qquad (22)$$

After substituting (21) into (22), we arrive at an equation for determining the radius $r'=R'_{\inf}$ of the circle, on which the inflection points of the streamline family are located in the considered spiral flow. The equation was solved numerically using the secant method. Graphs in Fig. 4 present the type of relation to the Reynolds number of the root of this equation defining, with consideration to (21), an inflection point position on streamline.

In the range of values of the radial coordinate $R'_{\inf}<r'<R'_2$, streamline monotonically moves away from the radial axis. By contrast, after the passing of the inflection point in the range of $1<r'<R'_{\inf}$ the streamline deviates in the other direction and begins to twist around the axis of symmetry up to the exit through the inner boundary of the cylindrical layer.

Figure 5 shows the pressure distributions in a small vicinity of the exit from the cylindrical layer with $I'_{2b}=1{,}01$ and $P'_2=50$, calculated by the formula (12) and referred to corresponding pressure distribution (13) in the case of a purely radial flow ($I'_{2b}=1$):

$$\gamma'_p\left(r'\right)=P'\left(r'\right)/P'_{rad}\left(r'\right). \qquad (23)$$

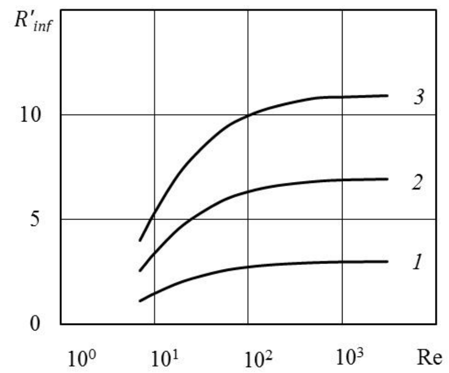

Fig. 4. Influence of the Reynolds number value on streamline inflection point with $I'_{2b} = 1{,}001$ and different values of R'_2: 3 (curve *1*); 7 (*2*); 11 (*3*).

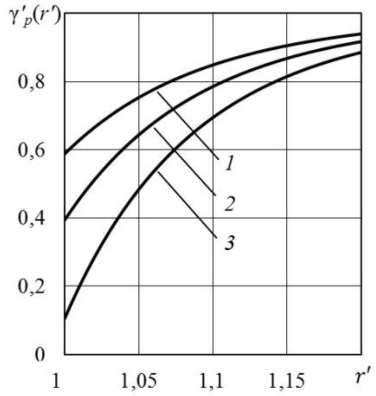

Fig. 5. Dependence of the relation (23) on the radial coordinate with $R'_2 = 3{,}6$ and different values of the Reynolds number Re: 6,2 (кривая *1*); 6,4 (*2*); 6,6 (*3*).

From the analysis of the dependences shown in Fig. 5, it follows that in a plane spiral flow, the pressure in a rather small vicinity of the exit from the cylindrical layer takes on lower values compared to the same pressure in a purely radial flow.

Considering the above mentioned specific feature of the pressure distribution in a plane spiral flow in the vicinity of the exit from the cylindrical layer and the fulfillment of the condition $P'(1) < P'_{rad}(1)$, let us make the following assumption: it is possible that the appearance of a natural tornado effect is a consequence of a decrease (its reasons are not discussed here) in pressure at the "flow" boundary (at the conditional boundaries of the central "trunk" of the tornado) below the level corresponding to the pressure in a purely radial flow pattern.

5. An alternative way of the notation of the second boundary condition for the azimuthal component of the velocity

It should be noted that the formulation of the boundary condition in the form (9) for the second invariant of the strain rate tensor is not traditional. This is due to the fact that the characteristic is, generally speaking, not entirely convenient from the point of view of its direct experimental determination and subsequent use. In this regard, let us consider a formally different, though essentially equivalent mode of writing the second boundary condition, which, taken together with the boundary condition (8), would make it possible to completely determine the integration constants C_1, C_2 in (7).

As follows from the behavior of function (23), a distinctive feature of a plane spiral flow is that the pressure $P'(1) = P'_1$ at the exit from the cylindrical layer takes on values less than the values obtained with other parameters being equal in the problem of purely radial flow. In this setting, pressure P'_1, defined from (12) at the exit from the cylindrical layer, is, in fact, determined by means of setting the value I'_{2b} at the entrance to the layer. In other words, there is a one-to-one correspondence between these quantities. In this concern, instead of a boundary condition (10) for a derivative of the function u'_φ along the radial coordinate at the entrance to the cylindrical layer, it is recommended to accept for pressure, but at the exit from the cylindrical layer, the following condition:

$$r' = 1: \qquad P' = P'_1 . \tag{24}$$

Here, naturally, it is assumed that P_1' is a given value that satisfies the condition $P_1' < P_{rad}'(1)$. Note also that (24) in this case is already, along with (4), the second boundary condition for pressure, despite the fact that differential equation (1) with respect to the function $P(r)$ has only the first order.

Then, taking into account the new boundary condition (24), the solution of the problem of a steady plane spiral flow in a cylindrical layer should be carried out in the following sequence:
- the expression for the radial velocity component is still determined by relation (5), since it follows from the continuity condition;
- taking into account the boundary condition (8), expression (7) for the azimuthal component of the flow velocity can be worked out to the form:

$$u_\varphi'(r') = \frac{C_1}{r'} \cdot \left[1 - \left(\frac{R_2'}{r'}\right)^{\mathrm{Re}-2}\right]; \tag{25}$$

- substitution of (5) and (25) into (1) and subsequent integration taking into account the boundary condition (4) leads to the following expression for the pressure distribution:

$$P'(r') = P_2' - f_1(r') + C_1^2 \cdot \left\{-f_1(r') + \frac{4 \cdot R_2'^{\mathrm{Re}-2}}{\mathrm{Re}} \cdot f_2(r') - \frac{R_2'^{2\mathrm{Re}-4}}{(\mathrm{Re}-1)} \cdot f_3(r')\right\}, \tag{26}$$

which, as expected, coincides structurally with (12);
- in accordance with the boundary condition (24), from (26) we derive the formula for determining the last unknown constant:

$$C_1 = \pm\sqrt{\frac{\left(P_2' - P_1' - f_1(1)\right) \cdot \mathrm{Re} \cdot (\mathrm{Re}-1)}{\mathrm{Re} \cdot (\mathrm{Re}-1) \cdot f_1(1) - 4(\mathrm{Re}-1) \cdot R_2'^{\mathrm{Re}-2} \cdot f_2(1) + \mathrm{Re} \cdot R_2'^{\mathrm{Re}-4} \cdot f_3(1)}}.$$

Thus, despite being alternative, the version of the second boundary condition in the form of (24) turns out to be equivalent to the boundary condition of the form (9) for determining the azimuthal component of the velocity. This is, as noted above, due to the fact that there is a one-to-one correspondence between the values I_{2b}' and P_1'.

In view of the last assertion, condition $P_1' \le P_{rad}'(1)$ leads directly to: $I_{2b}' \ge 1$, in the case of which boundary condition (10) makes sense. Indeed, considering (5), (25) and taking into account the boundary condition (8), we immediately obtain:

$$\left|I_2'(R_2')\right| = I_{2b}' = 1 + \frac{C_1^2 (\mathrm{Re}-2)^2}{4} \ge 1.$$

The condition accepted in the problem $P'(1) = P_1' < P_{rad}'(1)$ puts a restriction on the realization of a vortex flow in a cylindrical layer with a radial fluid supply through the outer surface: in order to make it possible, it is necessary to ensure the pressure at the exit of the layer below the level, which would take place here in the case of a purely radial flow with the same pressure value at the entrance to the layer. In its turn, to determine the value of P_1' (already beyond the scope of the given model problem), it would apparently be necessary to solve the vortex flow problem in a conjugate formulation and taking into account the three-dimensional nature of the fluid flow, including that in the interior area, where $r' < 1$.

6. Conclusion

The problem of a plane flow of the vortex flow type in a cylindrical layer has been considered. The assumption is that the fluid is supplied normally (radially) to the layer through its outer surface. As a basis, we have taken the general solution for the azimuthal component of the current velocity in unlimited space, obtained earlier in [40]. The final determination of the constants of integration of the known general solution as applied to the problem under discussion requires the formulation of two boundary conditions for the azimuthal component of the velocity. In such case, the form of at least one of them is not obvious. It has been suggested that such a boundary condition be represented in the form of a requirement to provide a given value of the second invariant of the strain rate tensor at the entrance to the cylindrical layer. As a result, this makes it possible to write down the exact solution of the problem under consideration, from which it follows, that even at very small excesses by the dimensionless parameter I'_{2b}, characterizing the value of the second invariant of the strain rate tensor at the entrance to the flow region, of threshold level $I'_{2b}=1$, a steady flow with streamlines such as plane spirals can be realized. It is shown that streamlines can have an inflection point.

Based on the obtained expression for the pressure distribution, it is shown that in the vicinity of the exit from the cylindrical layer, the pressure in the fluid takes values lower than in a purely radial flow. In view of this result, an alternative version of recording the second boundary condition is proposed for determining the azimuthal component of the velocity, which should be set for the pressure value at the exit of the cylindrical layer. Some consequences arising from the analysis of the considered vortex flow scheme, in the first approximation, may be of interest in modeling the corresponding hydrodynamic effects, such as a tornado or swirling flow in the vicinity of the drain hole. In this case, the results of the two-dimensional model problem solved in this work should be interpreted as the first approximation to the description of a more complex three-dimensional flow. In question here is a flow in a transverse (with respect to Oz axis of symmetry of a cylindrical coordinate system) section, provided that in the vicinity of this section the velocity components and pressure are weakly dependent on the z coordinate.

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