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UNSTEADY BENDING OF A CANTILEVERED EULER–BERNOULLI BEAM WITH DIFFUSION

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We considered the problem of unsteady direct bending of an isotropic homogeneous elastodiffusive cantilevered Euler–Bernoulli beam. For the mathematical formulation of the problem, we use the closed system of equations of transverse unsteady beam vibrations with inner diffusion. The formulation is based on the model of elastic diffusion using the d'Alember variational principle. It is assumed that the deflections of the beam are small and the hypothesis of flat sections is fulfilled. The Euler–Bernoulli hypothesis is valid for section rotations. A solution to the problem is sought using the method of equivalent boundary conditions, which allows a transition from the initial formulation with arbitrary boundary conditions to the problem of the same type and with the same domain geometry. First, an auxiliary problem is solved using the integral Laplace transform in time and trigonometric Fourier series. Then, some relations connecting the boundary conditions right-hand sides of the original and auxiliary problems are constructed. These relations are the Volterra integral equations of the first kind. For solving this system, quadrature formulas of an average rectangle are used. Finally, the solution of the original problem is represented in the form of the convolution of the Green's functions for the auxiliary problem with the functions determined by solving the system of the Volterra integral equations. The interaction between unsteady mechanical and diffusion fields is analyzed using an isotropic beam as an example. Graphs showing the dependence of the displacement fields and concentration increments on time and coordinates are given. Analysis of the results obtained led to conclusion that the coupling action of mechanical and diffusion fields affects the stress-strain state and mass transfer in a beam.

Key words*:* elastic diffusion, Green's function, Euler–Bernoulli beam, d'Alembert principle, equivalent boundary condition method, numerical analysis

1. Introduction

The effects of the different physical fields interaction are manifested in the form of mechanodiffusion, thermomechanodiffusion, electrodiffusion, magnetodiffusion, and others. These interactions have been well studied experimentally and are widely used in technology and industry. Various formulations and methods for solving the problems of mechanodiffusion with possible allowance for other physical fields have been actively considered in recent decades in the works of both Russian and foreign authors [1–4]. This indicates the relevance of research in this area.

On the other hand, it should be noted that most of the publications are related to modeling stationary and unsteady processes in canonical form bodies, in a layer or half-space. At the same time, real bodies have a finite size. hence the problems of mechanodiffusion in beams, plates, and shells, being the main elements of real structures and devices, are of great practical interest.

Relatively few publications are devoted to this topic, among which the work [5] can be noted. It evaluates the influence of diffusion processes on the bearing capacity of a shallow transversely isotropic shell. The contact interaction of a rod with an elastic half-space is discussed in articles [6, 7]. Publications [8–10] are devoted to the study of elastic diffusion processes in plates. The calculation of spherical shells with diffusion are considered in [11].

In the listed works, mechanodiffusion processes are stationary. Stationary processes are useful in the study of the steady-state operating conditions of various mechanical systems. It is necessary to use non-stationary models for the analysis of short-term and impulse effects. In publications [12–14] the authors modeled the mechanical and diffusion fields interaction caused by unsteady bending of simply supported beams and plates.

As is known, the boundary conditions significantly affect the approach to solving the initial-boundary value problem and the complexity of its implementation. For example, the conditions of a simply support make it possible to construct a solution in the series form by eigenfunctions of the corresponding elastic-diffusion operator. As for a cantilevered beam, which is considered in this paper, it is not possible to obtain such eigenfunctions. In this regard, the method of equivalent

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boundary conditions is applied. The essence of the method lies in the transition from the original problem to an auxiliary problem for the same research subject, but with boundary conditions that allow to represent the solution in the form of Fourier series. Further, relations are derived that connect the right-hand sides of the boundary conditions for the same parameters of both problems. They have the form of Volterra integral equations of the 1st kind. The system of these equations is solved numerically using quadrature formulas.

2. Formulation of the problem

The paper considers the unsteady elastic diffusion bending problem of a cantilevered homogeneous isotropic Bernoulli – Euler beam, a force being applied to the free end (Fig. 1).

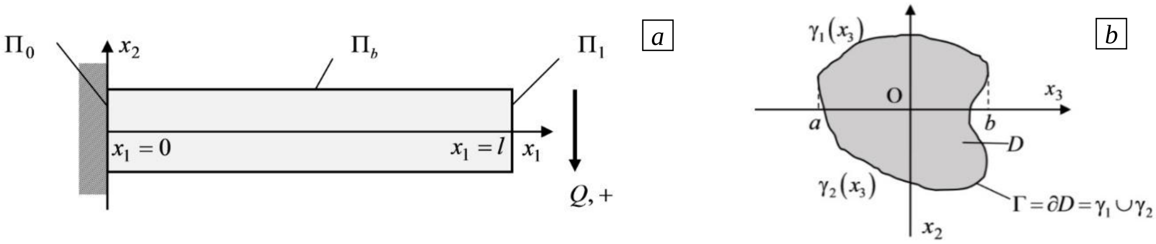

Fig. 1. Illustrations for the problem formulation: diagram of the applied forces (a) and the orientation of the coordinate axes in the cross-section of the beam (b).

The linear model of elastic diffusion in a continuum is used to describe the beam bending general case in a rectangular Cartesian coordinate system. If the deformable medium is homogeneous, the equations have the form [15–18]: $\ddot{u}_i = \dfrac{\partial \sigma_{ij}}{\partial x_j} + F_i, \quad \dot{\eta}^{(q)} = -\dfrac{\partial J_i^{(q)}}{\partial x_i} + Y^{(q)} \quad \left(i, j = \overline{1,3}, \quad q = \overline{1,N}\right),$ (1)

where the dots denote the time derivative; σ_{ij} and $J_i^{(q)}$ are the components of the stress tensor and the diffusion flux vector:

$$\sigma_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l} - \sum_{q=1}^{N} \alpha_{ij}^{(q)} \eta^{(q)}, \qquad J_i^{(q)} = -\sum_{t=1}^{N} D_{ij}^{(q)} g^{(qt)} \frac{\partial \eta^{(t)}}{\partial x_j} + \Lambda_{ijkl}^{(q)} \frac{\partial^2 u_k}{\partial x_j \partial x_l} \qquad \left(q = \overline{1,N}\right). \tag{2}$$

All quantities in (1) and (2) are dimensionless:

$$\begin{gathered} x_i = \frac{x^*_i}{l}, \quad u_i = \frac{u^*_i}{l}, \quad \tau = \frac{Ct}{l}, \quad C_{ijkl} = \frac{C^*_{ijkl}}{C_{1111}}, \quad C^2 = \frac{C^*_{1111}}{\rho}, \quad \alpha_{ij}^{(q)} = \frac{\alpha_{ij}^{*(q)}}{C_{1111}}, \\ D_{ij}^{(q)} = \frac{D_{ij}^{*(q)}}{Cl}, \quad \Lambda_{ijkl}^{(q)} = \frac{m^{(q)} D_{ij}^{*(q)} \alpha_{kl}^{*(q)} n_0^{(q)}}{\rho R T_0 C l}, \quad F_i = \frac{\rho l F_i^*}{C_{1111}}, \quad Y^{(q)} = \frac{l Y^{*(q)}}{C}, \end{gathered} \tag{3}$$

where the beam length l and the tension-compression wave velocity in a continuum C are used as characteristic scales. Also accepted designations: t is time; x_i^* are rectangular Cartesian coordinates; u_i^* are displacement vector components; $\eta^{(q)} = n^{(q)} - n_0^{(q)}$ is the concentration increment of q-th component in the multicomponent continuum; $n_0^{(q)}$ and $n^{(q)}$ is the initial and current concentrations (mass fractions) of q-th component; C_{ijkl}^* are components of the elastic constant tensor; ρ is the medium density; $\alpha_{ij}^{*(q)}$ are coefficients characterizing the medium volumetric changes due to diffusion; $D_{ij}^{*(q)}$ are the self-diffusion coefficients; R is the universal gas constant; T_0 is initial temperature; $m^{(q)}$ is the molar mass of q-th component; F_i^* are density of body forces; $Y^{*(q)}$ are density of bulk sources of mass transfer.

Equations (1) and (2) include N independent components describing the process of deformation of a multicomponent medium consisting of $N+1$-th substance. In this case, the increase or decrease in the mass fraction $N+1$-th is expressed through the increment in the fractions of the remaining N components of the medium:

$$\eta^{(N+1)} = -\sum_{q=1}^{N} \eta^{(q)},$$

which ensures the fulfillment of the mass conservation law:

$$\sum_{q=1}^{N+1} \eta^{(q)} = 0.$$

The initial and boundary conditions for the independent components of the equations in general form are written as:

$$u_i\Big|_{\tau=0} = u_{i0}, \quad \dot{u}_i\Big|_{\tau=0} = v_{i0}, \quad \eta^{(q)}\Big|_{\tau=0} = \eta_0^{(q)} \quad \left(i, j = \overline{1,3}, \quad q = \overline{1,N}\right), \tag{4}$$

$$u_i\Big|_{\Pi_u} = U_i, \quad \sigma_{ij} n_j\Big|_{\Pi_\sigma} = P_i \quad (\tau > 0), \quad \partial G = \Pi_u \cup \Pi_\sigma,$$
$$\eta^{(q)}\Big|_{\Pi_\eta} = N^{(q)}, \quad J_i^{(q)}\Big|_{\Pi_J} = I_i^{(q)} \quad \left(\tau > 0, \quad q = \overline{1,N}\right), \quad \partial G = \Pi_\eta \cup \Pi_J. \tag{5}$$

Here ∂G is area boundary G; n_i are components of the outer normal unit vector to ∂G; u_{i0}, v_{i0}, $\eta_0^{(q)}$ are given functions of spatial coordinates. Further we assume that $u_{i0} = 0$, $v_{i0} = 0$, $\eta_0^{(q)} = 0$. The quantities on the right-hand sides of the boundary conditions (5) are surface kinematic (U_i, $N^{(q)}$) and dynamic (P_i, $I_i^{(q)}$) perturbations.

To construct the beam bending equations, the following actions are taken:

– we pass to the variational formulation of problem (1), (2), (4), (5). According to the d'Alembert variational principle, these relations can be written as the following variational equation:

$$\int_G \left(\ddot{u}_i - \frac{\partial \sigma_{ij}}{\partial x_j} - F_i \right) \delta u_i dG + \sum_{q=1}^{N} \int_G \left(\dot{\eta}^{(q)} + \frac{\partial J_i^{(q)}}{\partial x_i} - Y^{(q)} \right) \delta \eta^{(q)} dG + \iint_{\Pi_\sigma} \left(\sigma_{ij} n_j - P_i \right) \delta u_i dS +$$
$$+ \sum_{q=1}^{N} \iint_{\Pi_J} \left(J_i^{(q)} - I_i^{(q)} \right) n_i \delta \eta^{(q)} dS = 0, \tag{6}$$

where δu_i are virtual displacement, $\delta \eta^{(q)}$ are virtual concentrations increments;

– the following assumptions are made:

1) The problem solution domain is cylinder $G = D \times [0,1]$, where D - area occupied by the beam cross-section. $\Gamma = \partial D = \gamma_1(x_3) \cup \gamma_2(x_3)$ is the section boundary (Fig. 1).

2) The axis Ox_3 is the central axis of the cross-section. In this case,

$$\iint_D x_2 dx_2 dx_3 = 0. \tag{7}$$

3) The surface of the beam is represented as $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_b$, where Π_0 is the surface corresponding $x_1 = 0$, Π_1 is the surface corresponding $x_1 = 1$, Π_b is the lateral surface. It is assumed that the side surface is free from mechanical loads, i.e.

$$\sigma_{ij} n_j\Big|_{\Pi_b} = 0. \tag{8}$$

We also assume that there is no mass transfer through the side surface:

$$J_i^{(q)}\Big|_{\Pi_b} = 0. \tag{9}$$

4) The beam material is a homogeneous isotropic continuum:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \quad \alpha_{ij}^{(q)} = \delta_{ij}\alpha_q, \quad D_{ij}^{(q)} = \delta_{ij}D_q, \quad \Lambda_{ijkl}^{(q)} = \delta_{ij}\delta_{kl}\Lambda_q, \tag{10}$$

where λ and μ are Lame coefficients, δ_{ij} is the Kronecker symbol. In this case, taking into account formulas (3), $\lambda + 2\mu = 1$.

5) From the standpoint of the mass transfer process, the beam material is perfect solid solution. In this case [15–18]:

$$g^{(qr)} = \delta_{qr}, \qquad D_{ij}^{(q)} g^{(qr)} = D_{ij}^{(q)} = \delta_{ij}D_q. \tag{11}$$

6) The beam bending is considered in plane Ox_1x_2. Then $u_k = u_k\left(x_1, x_2, \tau\right)$ $\left(k = 1,2\right)$ and $u_3 = 0$, $\varepsilon_{i3} = 0$ $\left(i = \overline{1,3}\right)$. Mass transfer occurs also in this plane: $\eta^{(q)} = \eta^{(q)}\left(x_1, x_2, \tau\right)$.

7) Transverse deflections are considered small. The cross-sections after deformation remain normal to the neutral line of the beam (the plane-sections hypothesis). Then the sought quantities $u_1\left(x_1, x_2, \tau\right)$, $u_2\left(x_1, x_2, \tau\right)$ and $\eta^{(q)}\left(x_1, x_2, \tau\right)$ linearized with respect to the variable x_2 can be represented as [12–14]:

$$\begin{gathered} u_1\left(x_1, x_2, \tau\right) = u\left(x_1, \tau\right) + x_2\chi\left(x_1, \tau\right), \qquad u_2\left(x_1, x_2, \tau\right) = v\left(x_1, \tau\right) + x_2\psi\left(x_1, \tau\right), \\ \eta^{(q)}\left(x_1, x_2, \tau\right) = N_q\left(x_1, \tau\right) + x_2 H_q\left(x_1, \tau\right). \end{gathered} \tag{12}$$

8) Due to the accepted Bernoulli – Euler hypothesis, the cross sections after deformation remain normal to the curved axis of the beam. In addition, when the lateral surface is free from loads, it can be assumed that deformations along the axis Ox_2 are absent due to their smallness. Then [12, 14] (the prime denotes the derivative with respect to the variable x_1)

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \psi = 0 \Rightarrow \psi = 0, \quad \chi\left(x_1, \tau\right) = -v'\left(x_1, \tau\right),$$

and equalities (12) are written as:

$$\begin{gathered} u_1\left(x_1, x_2, \tau\right) = u\left(x_1, \tau\right) - x_2 v'\left(x_1, \tau\right), \qquad u_2\left(x_1, x_2, \tau\right) = v\left(x_1, \tau\right), \\ \eta^{(q)}\left(x_1, x_2, \tau\right) = N_q\left(x_1, \tau\right) + x_2 H_q\left(x_1, \tau\right). \end{gathered} \tag{13}$$

Taking into account (10), (11), and (13), the components of the stress tensor and the diffusion flux vector will have the form:

$$\begin{gathered} \sigma_{11} = \left(u' - x_2 v''\right) - \sum_{q=1}^{N} \alpha_q\left(N_q + x_2 H_q\right), \quad \sigma_{22} = \lambda\left(u' - x_2 v''\right) - \sum_{q=1}^{N} \alpha_q\left(N_q + x_2 H_q\right), \quad \sigma_{12} = 0, \\ J_1^{(q)} = -D_q\left(N_q' + x_2 H_q'\right) + \Lambda_q\left(u'' - x_2 v'''\right), \quad J_2^{(q)} = -D_q H_q - \Lambda_q v'' \quad \left(q = \overline{1,N}\right). \end{gathered} \tag{14}$$

As a result, substituting equalities (7) - (14) into (6), we obtain the equations of elastic-diffusion vibrations of the beam [12]:

– longitudinal

$$\ddot{u} = u'' - \sum_{q=1}^{N} \alpha_q N_q' + \frac{n}{F}, \qquad \dot{N}_q = D_q N_q'' - \Lambda_q u''' + \frac{y^{(q)}}{F}; \tag{15}$$

– transverse

$$\ddot{v}'' - a\ddot{v} = v^{IV} + \sum_{j=1}^{N} \alpha_j H_j'' - \frac{q + m'}{F}, \qquad \dot{H}_q = D_q H_q'' + \Lambda_q v^{IV} + \frac{z^{(q)}}{J_3}, \qquad \frac{J_3}{F} = a. \tag{16}$$

In (15), (16), the notation is used: F is the cross-sectional area; J_3 are the moment of inertia of the beam section about the axis Ox_3; n is the linearly distributed axial load, m is the linearly distributed moment q is the linearly distributed transverse load; $y^{(q)}$ and $z^{(q)}$ is the linear density of bulk mass transfer sources.

We supplement the obtained equations (15), (16) with boundary conditions, which are also obtained from the variational equation (6). In accordance with the formulation of the original problem, the elastic diffusion mathematical model of the cantilevered beam bending under the action of a concentrated load at the free end includes equations (16) at $m = 0$, $q = 0$, $z^{(q)} = 0$ and the following boundary conditions:

$$\begin{gathered} v'\big|_{x=0} = 0, \quad v\big|_{x=0} = 0, \quad H_q\big|_{x=0} = 0, \quad \left(D_q H_q' + \Lambda_q v'''\right)\Big|_{x=1} = 0, \\ \left(v'' + \sum_{j=1}^{N} \alpha_j H_j'\right)\Bigg|_{x=1} = 0, \quad \left(v''' + \sum_{j=1}^{N} \alpha_j H_j' - \ddot{v}'\right)\Bigg|_{x=1} = f_{22}(\tau). \end{gathered} \tag{18}$$

Initial conditions will be assumed to be zero.

3. Method of solution

The main difficulty lies in the impossibility of constructing a solution to the above problem in the form of trigonometric Fourier series. This significantly complicates the inversion of the Laplace transform, which is also used to solve the problem. To overcome this problem, the method of equivalent boundary conditions is used [17, 18]. The method consists in performing the following series of steps.

First, instead of the original problem, we consider an auxiliary problem (16) with the boundary conditions:

$$\begin{gathered} v'\big|_{x=0} = 0, \quad v'\big|_{x=1} = f_{12}^{*}(\tau), \\ \left(D_q H_q' + \Lambda_q v'''\right)\Big|_{x=0} = f_{q+2,1}^{*}(\tau), \quad \left(D_q H_q' + \Lambda_q v'''\right)\Big|_{x=1} = 0, \\ \left(v''' + \sum_{j=1}^{N} \alpha_j H_j' - \ddot{v}'\right)\Bigg|_{x=0} = f_{21}^{*}(\tau), \quad \left(v''' + \sum_{j=1}^{N} \alpha_j H_j' - \ddot{v}'\right)\Bigg|_{x=1} = f_{22}(\tau), \end{gathered} \tag{19}$$

where functions $f_{12}^{*}(\tau)$, $f_{q+2,1}^{*}(\tau)$, $f_{21}^{*}(\tau)$ are to be determined. Taking into account (19), the solution takes the form [12]:

$$
\begin{gathered}
v(x,\tau)=\int_0^\tau\left[G_{12}(x,\tau-t)\,f_{21}^*(t)-G_{12}(1-x,\tau-t)\,f_{22}(t)\right]dt-\int_0^\tau G_{11}(1-x,\tau-t)\,f_{12}^*(t)\,dt+\\
+\sum_{p=1}^N\int_0^\tau G_{1,p+2}(x,\tau-t)\,f_{p+2,1}^*(t)\,dt,\\
\eta_q(x,\tau)=\int_0^\tau\left[G_{q+1,2}(x,\tau-t)\,f_{21}^*(t)-G_{q+1,2}(1-x,\tau-t)\,f_{22}(t)\right]dt-\int_0^\tau G_{q+1,1}(1-x,\tau-t)\,f_{12}^*(t)\,dt+\\
+\sum_{p=1}^N\int_0^\tau G_{q+1,p+2}(x,\tau-t)\,f_{p+2,1}^*(t)\,dt,
\end{gathered}
\tag{20}
$$

where G_{mk} are surface Green's functions of problem (16), (19). They are solutions to the following problems:

$$
\ddot{G}_{1k}''-a\ddot{G}_{1k}=G_{1k}^{IV}+\sum_{j=1}^N\alpha_jG_{j+1,k}'',\qquad \dot{G}_{j+1,k}=D_qG_{q+1,k}''+\Lambda_qG_{1k}^{IV};
\tag{21}
$$

$$
\begin{gathered}
\left.\left(G_{1k}'''+\sum_{j=1}^N\alpha_jG_{j+1,k}'-\ddot{G}_{1k}'\right)\right|_{x_1=0}=\delta_{2k}\delta(\tau),\quad G_{1k}'\big|_{x_1=0}=\delta_{1k}\delta(\tau),\quad \left.\left(G_{1k}'''+\sum_{j=1}^N\alpha_jG_{j+1,k}'-\ddot{G}_{1k}'\right)\right|_{x_1=1}=0,\\
G_{1k}'\big|_{x_1=1}=0,\quad \left.\left(D_qG_{q+1,k}'+\Lambda_qG_{1k}'''\right)\right|_{x_1=0}=\delta_{q+2,k}\delta(\tau),\quad \left.\left(D_qG_{q+1,k}'+\Lambda_qG_{1k}'''\right)\right|_{x_1=1}=0.
\end{gathered}
\tag{22}
$$

To find the Green's functions, the Laplace transform in time and the expansion in trigonometric Fourier series are used. After applying the indicated actions to problem (21), (22), a system of linear algebraic equations is obtained (subscript L means Laplace transform, $\lambda_n=\pi n$, $n=0,1,2,\ldots$):

$$
\begin{gathered}
k_1(\lambda_n,s)G_{1k}^{Lc}(\lambda_n,s)-\lambda_n^2\sum_{j=1}^N\alpha_jG_{j+1,k}^{Lc}(\lambda_n,s)=F_{1k}(\lambda_n),\qquad k_{q+1}(\lambda_n,s)G_{q+1,k}^{Lc}(\lambda_n,s)-\Lambda_q\lambda_n^4G_{1k}^{Lc}(\lambda_n,s)=F_{q+1,k}(\lambda_n),\\
G_{mk}^L(x,s)=\frac{G_{mk}^{Lc}(0,s)}{2}+\sum_{n=1}^\infty G_{mk}^{Lc}(\lambda_n,s)\cos\lambda_nx,\quad G_{mk}^{Lc}(\lambda_n,s)=2\int_0^1G_{mk}^L(x,s)\cos\lambda_nx\,dx,\\
k_1(\lambda_n,s)=\left(\lambda_n^2+a\right)s^2+\lambda_n^4,\qquad k_{q+1}(\lambda_n,s)=s+D_q\lambda_n^2,\\
F_{1k}(\lambda_n)=-2\lambda_n^2\delta_{1k}+2\delta_{2k},\quad F_{1k}(0)=\delta_{2k},\quad F_{q+1,k}(\lambda_n)=2\lambda_n^2\Lambda_q\delta_{1k}-2\delta_{q+1,k},\quad F_{q+1,k}(0)=-\delta_{q+1,k}.
\end{gathered}
$$

The solution of this system is found by Cramer's formulas and has the form:

$$
\begin{gathered}
G_{1k}^{Lc}(0,s)=\frac{\delta_{2k}}{as^2},\qquad G_{q+1,k}^{L}(0,s)=-\frac{\delta_{q+2,k}}{s},\\
G_{1k}^{Lc}(\lambda_n,s)=\frac{P_{1k}(\lambda_n,s)}{P(\lambda_n,s)},\qquad G_{q+1,2}^{Lc}(\lambda_n,s)=\frac{P_{q+1,2}(\lambda_n,s)}{Q_q(\lambda_n,s)},\\
G_{q+1,1}^{Lc}(\lambda_n,s)=\frac{2\Lambda_q\lambda_n^2}{k_{q+1}(\lambda_n,s)}+\frac{P_{q+1,1}(\lambda_n,s)}{Q_q(\lambda_n,s)},\qquad G_{q+1,p+2}^{Lc}(\lambda_n,s)=\frac{-2\delta_{qp}}{k_{q+1}(\lambda_n,s)}+\frac{P_{q+1,p+2}(\lambda_n,s)}{Q_q(\lambda_n,s)},
\end{gathered}
\tag{23}
$$

where

$$P(\lambda_n,s)=k_1(\lambda_n,s)\Pi(\lambda_n,s)-\lambda_n^6\sum_{j=1}^{N}\alpha_j\Lambda_j\Pi_j(\lambda_n,s),\qquad Q_q(\lambda_n,s)=k_{q+1}(\lambda_n,s)P(\lambda_n,s),$$

$$P_{11}(\lambda_n,s)=-2\lambda_n^2\left[\Pi(\lambda_n,s)+\lambda_n^2\sum_{j=1}^{N}\alpha_j\Lambda_j\Pi_j(\lambda_n,s)\right],$$

$$P_{12}(\lambda_n,s)=2\Pi(\lambda_n,s),\qquad P_{1,q+2}(\lambda_n,s)=-2\alpha_q\lambda_n^2\Pi_q(\lambda_n,s),\qquad P_{q+1,k}(\lambda_n,s)=\Lambda_q\lambda_n^4P_{1k}(\lambda_n,s),$$

$$\Pi(\lambda_n,s)=\prod_{j=1}^{N}k_{q+1}(\lambda_n,s),\qquad \Pi_q(\lambda_n,s)=\prod_{j=1,\,j\neq q}^{N}k_{j+1}(\lambda_n,s).$$

Since the obtained solutions (23) are rational functions of the Laplace transform parameter s, their originals are found using residues and tables of operational calculus [19]:

$$G_{1k}^c(0,\tau)=\frac{\delta_{2k}\tau}{a},\qquad G_{q+1,k}^c(0,\tau)=-\delta_{q+2,k}H(\tau),\qquad G_{1k}^c(\lambda_n,\tau)=\sum_{j=1}^{N+2}A_{1k}^{(j)}(\lambda_n)e^{s_j(\lambda_n)\tau},$$

$$G_{q+1,k}^c(\lambda_n,\tau)=2\left(\Lambda_q\lambda_n^2\delta_{1k}-\delta_{q+2,k}\right)e^{-D_q\lambda_n^2\tau}+\sum_{l=1}^{N+3}A_{q+1,k}^{(l)}(\lambda_n)e^{s_l(\lambda_n)\tau},\tag{24}$$

$$A_{1k}^{(j)}(\lambda_n)=\frac{P_{1k}(\lambda_n,s_j)}{P'(\lambda_n,s_j)},\qquad A_{q+1,k}^{(l)}(\lambda_n)=\frac{P_{q+1,k}(\lambda_n,s_l)}{Q_q'(\lambda_n,s_l)}.$$

Here: $H(\tau)$ is the Heaviside step function; $s_j(\lambda_n)$ $\left(j=\overline{1,\,N+2}\right)$ are the zeroes of polynomial $P(\lambda_n,s)$; $s_{N+3}(\lambda_n)=-D_q\lambda_n^2$ are the additional zero of the polynomial $Q_q(\lambda_n,s)$; the prime denotes the derivative with respect to the parameter s.

Further, relations are constructed connecting the boundary conditions right-hand sides of original problem and auxiliary problem. The solution to the auxiliary problem (16), (19) must satisfy the boundary conditions of the original problem for the cantilevered beam (18). Then, taking into account representations (20), the relation expressions can be written in a system of Volterra integral equations of the 1st kind [17, 18]:

$$\sum_{j=1}^{N+2}\int_0^\tau a_{ij}(\tau-t)y_j(t)dt=\varphi_i(\tau),\tag{25}$$

where (the prime denotes the derivative with respect to the spatial variable x)

$$a_{11}(\tau)=G_{12}(0,\tau),\qquad a_{12}(\tau)=-G_{11}(1,\tau),\qquad a_{1,p+2}(\tau)=G_{1,p+2}(0,\tau),$$

$$a_{21}(\tau)=G_{12}''(1,\tau)+\sum_{j=1}^{N}\alpha_jG_{j+1,2}(1,\tau),\qquad a_{22}(\tau)=-G_{11}''(0,\tau)-\sum_{j=1}^{N}\alpha_jG_{j+1,1}(0,\tau),$$

$$a_{2,q+2}(\tau)=G_{1,q+2}''(1,\tau)+\sum_{j=1}^{N}\alpha_jG_{j+1,q+2}(1,\tau),$$

$$a_{q+2,1}(\tau)=G_{q+1,2}(0,\tau),\qquad a_{q+2,2}(\tau)=-G_{q+1,1}(1,\tau),\qquad a_{q+2,p+2}(\tau)=G_{q+1,p+2}(0,\tau),$$

$$y_1(\tau)=f_{21}^*(\tau),\qquad y_2(\tau)=f_{12}^*(\tau),\qquad y_{p+2}(\tau)=f_{p+2,1}^*(\tau),$$

$$\varphi_1(\tau)=\int_0^\tau G_{12}(1,\tau-t)f_{22}(t)dt,\qquad \varphi_{q+2}(\tau)=\int_0^\tau G_{q+1,2}(1,\tau-t)f_{22}(t)dt,$$

$$\varphi_2(\tau)=\int_0^\tau\left[G_{12}''(0,t-\tau)+\sum_{j=1}^{N}\alpha_jG_{j+1,2}(0,\tau-t)\right]f_{22}(t)dt.$$

It should be noted that in accordance with formulas (23), (24) the Fourier series in the representations for G''_{km} converge only in a generalized sense. This complicates the application of numerical algorithms to solve the system (25). To overcome this difficulty, let us integrate the integrals in (25) by parts. We obtain the system of equations for the derivative $\partial y_j / \partial \tau$:

$$\sum_{j=1}^{N+2} \int_0^\tau A_{ij}(\tau - t) \frac{\partial y_j(t)}{\partial t} dt = \Phi_i(\tau), \quad \Phi_i(\tau) = \varphi_i(\tau) - \sum_{j=1}^{N+2} A_{ij}(\tau) y_j(0), \tag{26}$$

$$A_{ij}(\tau) = \int_0^\tau a_{ij}(t)\, dt, \quad A_{ij}(\tau - t) = \int_0^{\tau - t} a_{ij}(\xi)\, d\xi. \tag{27}$$

In this case, the functions $y_j(0)$ should satisfy certain relations. Based on the condition of initial and boundary conditions conjugation at the space-time domain corner points of the considered problems, and also taking into account the zero initial conditions, we will further assume that $y_j(0) = 0$.

The system of equations (26) is solved numerically using quadrature formulas. For this, the region of time variation τ $[0, T]$ is divided with a uniform step $h = T/N_\tau$ into N_τ segments and grid functions are introduced $y_m^j = \partial y_j(\tau_m)/\partial \tau$, $A_m^{ij} = A_{ij}(\tau_m)$ at points $\tau_m = mh$ $\left(m = \overline{0, N_\tau}\right)$. Each of the integrals in (26) at $\tau = \tau_m$ is replaced by an approximate sum corresponding to the midpoint rectangles formula:

$$\int_0^\tau A_{ij}(\tau - t) \frac{\partial y_j(t)}{\partial t} dt \approx h S_{m-1/2}^{ij} + h A_{1/2}^{ij} y_{m-1/2}^j, \quad S_{m-1/2}^{ij} = \sum_{l=1}^{m-1} A_{m-l+1/2}^{ij} y_{l-1/2}^j \quad \left(i, j = \overline{1, N+2}\right),$$

$$\tau_{m-1/2} = \frac{\tau_{m-1} + \tau_m}{2} = h\left(m - \frac{1}{2}\right), \quad \tau_{m-l+1/2} = \tau_m - \tau_{l-1/2} = h\left(m - l + \frac{1}{2}\right) \quad \left(m = \overline{1, N_\tau}\right).$$

As a result, we get a recurrent sequence of systems of linear algebraic equations $(m \geq 1)$:

$$\mathbf{A}\mathbf{y}_{m-1/2} = \mathbf{b}_{m-1/2},$$

where $\mathbf{y}_{m-1/2} = \left(y_{m-1/2}^i\right)_{(N+2)\times 1}$ is the column of unknowns, and the remaining value are determined as follows:

$$\mathbf{A} = \left(A_{1/2}^{ij}\right)_{(N+2)\times(N+2)}, \quad \mathbf{b}_{m-1/2} = \left(b_{m-1/2}^i\right)_{(N+2)\times 1}, \quad b_{m-1/2}^i = \frac{\Phi_i(\tau_m)}{h} - \sum_{j=1}^{N+2} S_{m-1/2}^{ij}.$$

Its solution is found by the Cramer's formulas:

$$y_{m-1/2}^j = \frac{\Delta_m^j}{\Delta},$$

where $\Delta = \det \mathbf{A}$, Δ_m^j are the Cramer determinants for matrices that are obtained from matrix $\mathbf{A}$ by replacing the j-th column with a column $\mathbf{b}_{m-1/2}$.

The final form of the solution to the original problem is found by numerically calculating the convolutions of the Green's functions of auxiliary problems (21), (22) with functions determined from the solution of the system of equations (26). Thus, equations (20) can be written as:

$$v(x,\tau)=\int_0^\tau\left[\tilde{G}_{12}(x,\tau-t)\frac{f_{21}^*(t)}{\partial t}-\tilde{G}_{12}(1-x,\tau-t)\frac{f_{22}(t)}{\partial t}-\tilde{G}_{11}(1-x,\tau-t)\frac{f_{12}^*(t)}{\partial t}\right]dt+$$
$$+\int_0^\tau\sum_{p=1}^N\tilde{G}_{1,p+2}(x,\tau-t)\frac{f_{p+2,1}^*(t)}{\partial t}dt,$$
$$\eta_q(x,\tau)=\int_0^\tau\left[\tilde{G}_{q+1,2}(x,\tau-t)\frac{f_{21}^*(t)}{\partial t}-\tilde{G}_{q+1,2}(1-x,\tau-t)\frac{f_{22}(t)}{\partial t}-\tilde{G}_{q+1,1}(1-x,\tau-t)\frac{f_{12}^*(t)}{\partial t}\right]dt+ \quad (28)$$
$$+\int_0^\tau\sum_{p=1}^N\tilde{G}_{q+1,p+2}(x,\tau-t)\frac{f_{p+2,1}^*(t)}{\partial t}dt,$$
$$\tilde{G}_{mk}(x,\tau)=\int_0^\tau G_{mk}(x,t)dt.$$

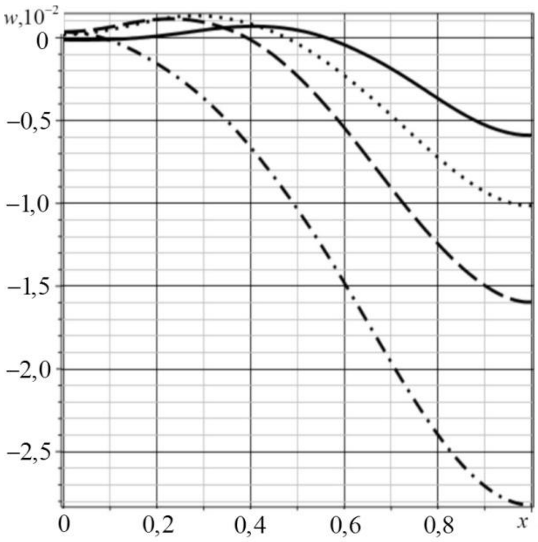

Fig. 2. Beam deflections $v(x,\tau)$ at different points in time τ: 3,3 (solid line); 5 (dotted line); 6,6 (dashed line); 10 (dash-dotted line).

4. Example

To study the interaction of unsteady fields (mechanical and diffusion), a cantilevered beam with a length of $l=1$ cm with a rectangular section $h\times b=0,05l\times 0,05l$, made of a 2-component material with 95% aluminum and 5% copper, is considered. Copper acts as an independent component [20]. For the calculation, the following material parameters were set (λ^*, μ^* are dimensional analogs of the Lame coefficients; superscript 1 indicates copper characteristics):

$$\lambda^*=6,93\cdot 10^{10}\text{ N/m}^2,\quad \mu^*=2,56\cdot 10^{10}\text{ N/m}^2,$$
$$T_0=800\text{ K},\quad \rho=2780\text{ kg/m}^3,\quad l=0,01\text{ m},$$
$$D_{11}^{*(1)}=6,67\cdot 10^{-14}\text{ m}^2\text{/sec},\quad n_0^{(1)}=0,05,$$

$$\alpha_{11}^{*(1)}=6,14\cdot 10^7\text{ J/kg},\quad m^{(1)}=0,064\text{ kg/mol}$$

The transverse load applied to the end of the beam ($x=1$) was set in the form:

$$f_{22}(\tau)=H(\tau).$$

Numerical solution of system (26) and substitution of the found functions into convolutions (28) results in beam deflections. The graphical solution is shown in Figure 2.

Figure 3 shows how the concentration distributions of copper (Fig.3a) and aluminum (Fig.3b) change as a result of unsteady bending of a cantilevered beam.

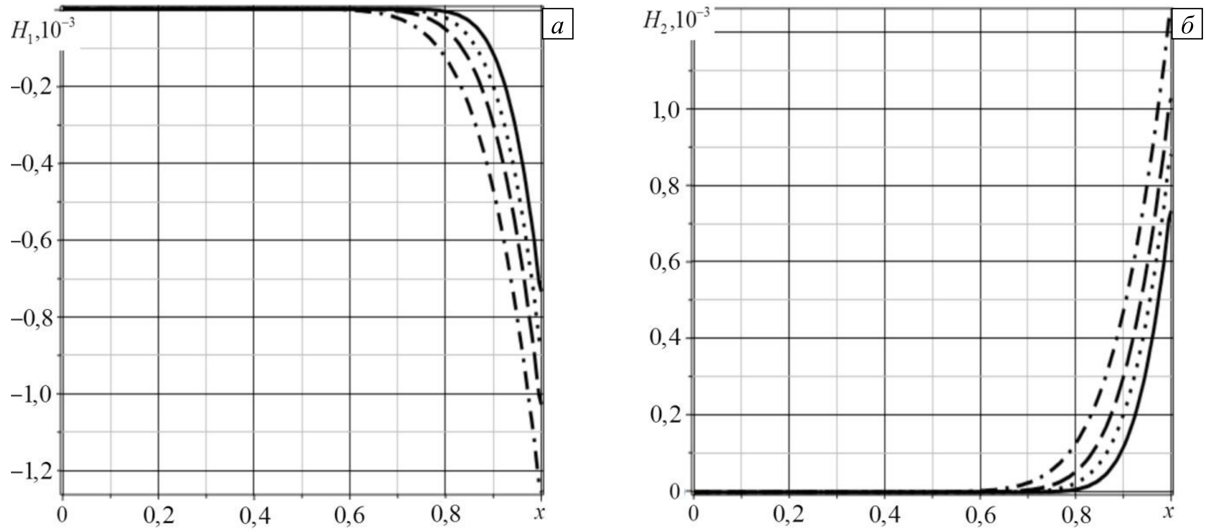

Fig. 3. Linear density of copper (a) and aluminum (b) concentration increment at different points in time τ : $3{,}3 \cdot 10^{12}$ (solid line); $5 \cdot 10^{12}$ (dotted line); $6{,}6 \cdot 10^{12}$ (dashed line); 10^{13} (dash-dotted line).

5. Conclusion

Thus, an unsteady bending model of a cantilevered Bernoulli–Euler beam with elastic diffusion effects is constructed. An algorithm is proposed that allows one to express the solution of a problem with arbitrary boundary conditions in terms of a known solution to a problem of the same type and with the same domain geometry.

The capabilities of the model and the algorithm are demonstrated by the example of calculating the bending of a two-component beam. It is shown that the unsteady bending of the cantilevered beam initiates diffusion fluxes of each of the components (Fig. 3). Considering the fact that at $N_q(x_1,\tau)=0$ the concentration increments are determined by formulas (13), the following conclusion can be drawn. During deformation in the lower part of the beam (in accordance with Fig. 1), the increase in copper concentration has a negative value, and in the upper part, it is positive. The result is an upward diffusion flux of copper, which is compensated by a downward flux of aluminum particles. In this case, the magnitude of the diffusion flux increases from the pinching of the beam to the end free from the load.

A certain drawback of the proposed algorithm is that the constructed system of Volterra equations connecting the right-hand sides of the boundary conditions of the original and auxiliary problems has to be solved numerically. A stable result when determining displacements can be obtained only for relatively short periods of time ($\tau \le 10$, Fig. 2). An increase in the duration of the interval leads to the need to use a larger number of partitions, which complicates the implementation of the proposed algorithm. At the same time, it should be noted that the results shown in Figure 2 correspond to the classical concepts of the cantilevered beams bending [21].

In contrast, due to the rather slow flux of diffusion in solids, it is possible to calculate the increments in the concentrations of a multicomponent medium at sufficiently long time intervals ($\tau \sim 10^{12} \div 10^{13}$) with a relatively small number of partition points.

In the numerical solution of the system of Volterra equations, in both cases, the partition $N_\tau = 40$ points were used. A further increase in their number no longer led to a visible change in the results.

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