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ON ONE VERSION OF THE GODUNOV METHOD FOR CALCULATING ELASTOPLASTIC DEFORMATIONS OF A MEDIUM

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Godunov's hybrid method suitable for numerical calculation of elastoplastic deformation of a solid body within the framework of the classical Prandtl-Reis model with the non-barotropic state equation is described. Mises' fluidity condition is used as a criterion for the transition from elastic to plastic state. A characteristic analysis of the model equations was carried out and their hyperbolicity was shown. It is noted that, if one takes the Maxwell-Cattaneo law instead of the Fourier law, then the Godunov hybrid method can be applied to calculate the deformation of a thermally conductive elastoplastic medium, since in this case the medium model is of a hyperbolic type. The algorithm for solving the systems in which there are equations that do not lead to divergence form is described in detail; Godunov's original method serves to integrate systems of equations represented in divergence form. When calculating stream variables on the faces of adjacent cells, a linearized Riemannian solver is used, the algorithm of which includes the right eigenvectors of the model equations. In the proposed approach, the equations written in divergence form look like finite-volume formulas, and others that do not lead to divergence form look like finite-difference relations. To illustrate the capabilities of the Godunov hybrid method, several one- and two-dimensional problems were solved, in particular, the problem of hitting an aluminum sample against a rigid barrier. It is shown that, depending on the rate of interaction, either single-wave or two-wave reflections described in the literature can be implemented with an elastic precursor.

***Key words**:* elastic-plastic deformations, hybrid Godunov method, Riemann linearized solver

1. Iintroduction

Examples of using the Godunov method [1–3], originally proposed by Godunov for numerical integration of gas-dynamic problems [4], in modeling elastoplastic deformation of the medium are known from literary sources. So, in [1] according to Godunov's scheme of the first order of accuracy, flat and axisymmetric problems of the dynamics of compressed media with irreversible volumetric and shear deformations on arbitrary movable grids were solved; finite-difference relations were given, the problem of the decay of an arbitrary discontinuity was described in detail. In [2], UNO (Uniformly Non-Oscillatory)modification of the Godunov method, which has a second order of accuracy, was used for the medium in neglect of its final deformation. The gap decay problem, which is present in the algorithm of the method, was solved involving Riemann invariants. As in [1], work [2] considered the barotropic equation of state, so the energy equation was omitted. In [3], the spatial sampling of equations on a moving Eulerian grid was carried out using the Godunov method. To improve the accuracy of the scheme, piecemeal linear replenishment of grid functions was performed using an interpolation procedure of the MUSCL type, generalized to unstructured grids. The main idea of the approach was to split the system of defining equations into hydrodynamic and elastoplastic components. The equations of hydrodynamics were solved on the basis of a clearly implicit absolutely stable scheme, and material equations (elastoplastic component) using a two-step Runge-Kutta scheme.

In the present paper, the Godunov hybrid method (GHM) has been used for numerical integration of a hyperbolic system describing elastoplastic deformation of a solid with the equation of the state of the general non-barotropic form, which was previously applied by the author in numerical modeling of currents of heterogeneous mixtures [5]. Note that GHM is intended for integration hyperbolic systems in which there are equations written both in divergent form and not reduced in this form, while the original Godunov method assumes only the solution of equations represented in divergent form. Environment settings on the faces of adjacent cells were determined using a linearized Riemannian solver [6]. For the same purposes, the Riemann problem solver based on the characteristic approach can be used [7]. Since further in the work the integration areas are a combination of rectangular sub-areas, then a simplified version of GHM [8] will be considered

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when integrating the equations. In the presence of curvilinear sub-regions, it is necessary to subject to a finite-volume version of the GHM [9].

2. Characteristic analysis of model equations

Equations that describe the elastoplastic deformation of a solid within a model Prandtl-Reis [10] have the form

$$\begin{gathered}
\rho_{,t} + \left(\rho u_i\right)_{,x_i} = 0, \\
\left(\rho u_i\right)_{,t} + \left(\rho u_i u_j - \sigma_{ij}\right)_{,x_j} = 0, \\
\left(\rho e\right)_{,t} + \left(\rho u_j e - u_i \sigma_{ij}\right)_{,x_j} = 0, \\
s_{ij,t} + u_k s_{ij,x_k} - s_{ik}\omega_{jk} - s_{jk}\omega_{ik} + \lambda s_{ij} = 2\mu\left(e_{ij} - \frac{1}{3} e_{kk}\delta_{ij}\right), \\
\omega_{ij} = \frac{1}{2}\left(u_{i,x_j} - u_{j,x_i}\right), \quad e_{ij} = \frac{1}{2}\left(u_{i,x_j} + u_{j,x_i}\right),
\end{gathered} \tag{1}$$

Here: p – pressure; u_i are the components of the velocity vector; ρ is the density; ε and $e = \varepsilon + \frac{1}{2} u_i u_i$ – specific internal and total energy of the medium; σ_{ij} – stress tensor, which is presented in the form of a ball $p = -\frac{1}{3}\sigma_{ii}$ and a deviation s_{ij} parts $\sigma_{ij} = s_{ij} - p\delta_{ij}$, where δ_{ij} – is the Kronecker delta; e_{ij} – strain rate tensor; μ – shear modulus; $\lambda(t)$ – parameter; index after comma means differentiation by the corresponding variable.

To simplify the presentation, we will limit ourselves to a two-dimensional case. We rewrite system (1) in quasi-linear form, and instead of index symbols we use coordinate:

$$\begin{gathered}
\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \quad \frac{Du}{Dt} + \frac{1}{\rho}\left(\frac{\partial (p - s_{xx})}{\partial x} - \frac{\partial s_{xy}}{\partial y}\right) = 0, \quad \frac{Dv}{Dt} + \frac{1}{\rho}\left(\frac{\partial (p - s_{yy})}{\partial y} - \frac{\partial s_{xy}}{\partial x}\right) = 0, \\
\frac{Dp}{Dt} + \rho c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left[s_{xx}\frac{\partial u}{\partial x} + s_{yy}\frac{\partial v}{\partial y} + s_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] B = 0, \\
\frac{Ds_{xx}}{Dt} - 2s_{xy}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) - \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = -\lambda s_{xx}, \\
\frac{Ds_{xy}}{Dt} + \left(s_{xx} - s_{yy}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\lambda s_{xy}, \\
\frac{Ds_{yy}}{Dt} - 2s_{xy}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) = -\lambda s_{yy}.
\end{gathered} \tag{2}$$

Here $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$, $B = \left(\rho\frac{\partial \varepsilon}{\partial p}\right)^{-1}$, $c = \sqrt{B\left(\frac{p}{\rho} - \rho\frac{\partial \varepsilon}{\partial \rho}\right)}$ – is the hydrodynamic speed of sound in a solid with a calorical equation of state of general form $\varepsilon = \varepsilon(p, \rho)$. Note that under the condition $s_{xx} = s_{xy} = s_{yy} = \mu = 0$, the solid behaves like a compressible liquid, since in this case the system (2) coincides with the equations of hydrodynamics. As a criterion for the transition from elastic to plastic co-standing, the Mises flow condition was used, according to which, if there is inequality

$$s_{xx}^2 + s_{xy}^2 + s_{yy}^2 + s_{xx}s_{yy} \ge \frac{1}{3}\sigma_s^2,$$

where σ_s – is the yield strength of the material at uniaxial tension-compression, then the components of the stress tensor deviator are corrected by "fit" on the yield surface – division by $\sqrt{\lambda}$ [11]. Parameter λ characterizes the total operation of plastic deformations and is calculated by the formula

$$\lambda = \frac{3}{\sigma_s^2}\left(s_{xx}^2 + s_{xy}^2 + s_{yy}^2 + s_{xx}s_{yy}\right).$$

The system (2) is rewritten in vector-matrix form

$$\frac{\partial \mathbf{U}}{\partial t} + A_x \frac{\partial \mathbf{U}}{\partial x} + A_y \frac{\partial \mathbf{U}}{\partial y} = \mathbf{H}, \tag{3}$$

where

$$\mathbf{U} = \left(\rho, u, v, p, s_{xx}, s_{xy}, s_{yy}\right)^{\mathrm{T}}, \quad \mathbf{H} = -\lambda\left(0,0,0,0,s_{xx}, s_{xy}, s_{yy}\right)^{\mathrm{T}},$$

$$A_x = \begin{pmatrix} u & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} & -\frac{1}{\rho} & 0 & 0 \\ 0 & 0 & u & 0 & 0 & -\frac{1}{\rho} & 0 \\ 0 & \rho c^2 - s_{xx}B & -s_{xy}B & u & 0 & 0 & 0 \\ 0 & -\frac{4}{3}\mu & 2s_{xy} & 0 & u & 0 & 0 \\ 0 & 0 & s_{xx} - s_{yy} - \mu & 0 & 0 & u & 0 \\ 0 & \frac{2}{3}\mu & -2s_{xy} & 0 & 0 & 0 & u \end{pmatrix}, \quad A_y = \begin{pmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & -\frac{1}{\rho} & 0 \\ 0 & 0 & v & \frac{1}{\rho} & 0 & 0 & -\frac{1}{\rho} \\ 0 & -s_{xy}B & \rho c^2 - s_{yy}B & v & 0 & 0 & 0 \\ 0 & -2s_{xy} & \frac{2}{3}\mu & 0 & v & 0 & 0 \\ 0 & s_{yy} - s_{xx} - \mu & 0 & 0 & 0 & v & 0 \\ 0 & 2s_{xy} & -\frac{4}{3}\mu & 0 & 0 & 0 & v \end{pmatrix}.$$

Here T is transposition operator. The eigenvalues of the A_x and A_y matrices are as follows:

$$\begin{gathered} u - c_{1x},\ u - c_{2x},\ u,\ u,\ u,\ \ u + c_{2x},\ u + c_{1x}, \\ v - c_{1y},\ v - c_{2y},\ v,\ v,\ v,\ \ v + c_{2y},\ v + c_{1y}, \end{gathered} \tag{4}$$

where

$$c_{1x} = \sqrt{c^2 + \frac{1}{\rho}\left(\frac{4}{3}\mu - s_{xx}B\right)}, \quad c_{2x} = \sqrt{\frac{1}{\rho}\left(\mu + s_{yy} - s_{xx}\right)}, \quad c_{1y} = \sqrt{c^2 + \frac{1}{\rho}\left(\frac{4}{3}\mu - s_{yy}B\right)},$$

$$c_{2y} = \sqrt{\frac{1}{\rho}\left(\mu + s_{xx} - s_{yy}\right)}.$$

Thus, in a solid body there are two speeds of sound: the first c_1 is elastic; the second c_2 is shear. Eigenvectors – columns of the underlying matrices

$$\begin{pmatrix} \frac{3\rho}{2\mu} & \frac{3\rho(B+2)}{2N_x} & 0 & 0 & 1 & \frac{3\rho(B+2)}{2N_x} & \frac{3\rho}{2\mu} \\ -\frac{3c_{1x}}{2\mu} & -\frac{3c_{2x}(B+2)}{2N_x} & 0 & 0 & 0 & \frac{3c_{2x}(B+2)}{2N_x} & \frac{3c_{1x}}{2\mu} \\ 0 & -\frac{3\rho c_{2x}\left(c_{1x}^2-c_{2x}^2\right)}{2s_{xy}N_x} & 0 & 0 & 0 & \frac{3\rho c_{2x}\left(c_{1x}^2-c_{2x}^2\right)}{2s_{xy}N_x} & 0 \\ \frac{3\rho c_{1x}^2}{2\mu}-2 & \frac{L_x}{2N_x} & 0 & 1 & 0 & \frac{L_x}{2N_x} & \frac{3\rho c_{1x}^2}{2\mu}-2 \\ -2 & \frac{M_x}{N_x} & 0 & 1 & 0 & \frac{M_x}{N_x} & -2 \\ 0 & -\frac{3\rho^2 c_{2x}^2\left(c_{1x}^2-c_{2x}^2\right)}{2s_{xy}N_x} & 0 & 0 & 0 & -\frac{3\rho^2 c_{2x}^2\left(c_{1x}^2-c_{2x}^2\right)}{2s_{xy}N_x} & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{3\rho}{4\mu} & -\frac{3\rho(B+2)}{2N_y} & 0 & 0 & 1 & -\frac{3\rho(B+2)}{2N_y} & -\frac{3\rho}{4\mu} \\ 0 & -\frac{3\rho c_{2y}\left(c_{1y}^2-c_{2y}^2\right)}{2s_{xy}N_y} & 0 & 0 & 0 & \frac{3\rho c_{2y}\left(c_{1y}^2-c_{2y}^2\right)}{2s_{xy}N_y} & 0 \\ \frac{3c_{1y}}{4\mu} & -\frac{3c_{2y}(B+2)}{2N_y} & 0 & 0 & 0 & \frac{3c_{2y}(B+2)}{2N_y} & -\frac{3c_{1y}}{4\mu} \\ 1-\frac{3\rho c_{1y}^2}{4\mu} & \frac{L_y}{2N_y} & 1 & 0 & 0 & \frac{L_y}{2N_y} & 1-\frac{3\rho c_{1y}^2}{4\mu} \\ -\frac{1}{2} & \frac{M_y}{N_y} & 0 & 1 & 0 & \frac{M_y}{N_y} & -\frac{1}{2} \\ 0 & \frac{3\rho^2 c_{2y}^2\left(c_{1y}^2-c_{2y}^2\right)}{2s_{xy}N_y} & 0 & 0 & 0 & \frac{3\rho^2 c_{2y}^2\left(c_{1y}^2-c_{2y}^2\right)}{2s_{xy}N_y} & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix},\tag{5}$$

where

$$L_x = 6\rho c^2 + \left(3\rho c_{2x}^2 - 6s_{xx} - 4\mu\right)B, \quad M_x = 3\rho\left(c_{1x}^2 - c_{2x}^2\right) - 2\mu(B+2),$$
$$N_x = 3\rho\left(c_{2x}^2 - c^2\right) + \left(3s_{xx} + \mu\right)B - 2\mu, \quad L_y = -6\rho c^2 - \left(3\rho c_{2y}^2 - 6s_{yy} - 4\mu\right)B,$$
$$M_y = 3\rho\left(c_{1y}^2 - c_{2y}^2\right) - \mu(B+2), \quad N_y = 3\rho\left(c_{2y}^2 - c^2\right) + \left(3s_{yy} + 2\mu\right)B,$$

corresponding to eigenvalues of matrices A_x and A_y are linearly independent, therefore the system of equations (2) under consideration belongs to hyperbolic type [12].

The GHM discussed in the work can also be used to calculate the deformation of a thermally conductive elastoplastic medium. For this case, the energy conservation law equation of (2) takes the form

$$\frac{Dp}{Dt} + \rho c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\rho\frac{\partial \varepsilon}{\partial p}\right)^{-1}\left[s_{xx}\frac{\partial u}{\partial x} + s_{yy}\frac{\partial v}{\partial y} + s_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - \frac{\partial w_x}{\partial x} - \frac{\partial w_y}{\partial y}\right] = 0,$$

where w_x and w_y projections of heat flux density vector on coordinate axes. In order to remain in the framework of the hyperbolic model, instead of the Fourier law, it is necessary to use its hyperbolic analogue, that is, the Maxwell-Cattaneo law [13]:

$$\tau \frac{Dw_x}{Dt} + w_x = -k \frac{\partial T}{\partial x}, \quad \tau \frac{Dw_y}{Dt} + w_y = -k \frac{\partial T}{\partial y}, \tag{6}$$

where T – is temperature, τ – relaxation time. Given the thermal equation of the state of the medium $T = T(p, \rho)$, as well as the expressions $\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial T}{\partial p}\frac{\partial p}{\partial x}$, $\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial T}{\partial p}\frac{\partial p}{\partial y}$, of equation (6) can be rewritten as

$$\frac{Dw_x}{Dt} + k_\rho \frac{\partial \rho}{\partial x} + k_p \frac{\partial p}{\partial x} = -\frac{w_x}{\tau}, \quad \frac{Dw_y}{Dt} + k_\rho \frac{\partial \rho}{\partial y} + k_p \frac{\partial p}{\partial y} = -\frac{w_y}{\tau},$$

where $k_\rho = \frac{1}{\tau}\frac{\partial T}{\partial \rho}$, $k_p = \frac{1}{\tau}\frac{\partial T}{\partial p}$. The corresponding matrices A_x and A_y of the modified model included in equation (3) take the form:

$$A_x = \begin{pmatrix} u & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} & -\frac{1}{\rho} & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 \\ 0 & \rho c^2 - s_{xx}B & -s_{xy}B & u & 0 & 0 & 0 & B & 0 \\ 0 & -\frac{4}{3}\mu & 2s_{xy} & 0 & u & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho c_{2x}^2 & 0 & 0 & u & 0 & 0 & 0 \\ 0 & \frac{2}{3}\mu & -2s_{xy} & 0 & 0 & 0 & u & 0 & 0 \\ k_\rho & 0 & 0 & k_p & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u \end{pmatrix}, \quad A_y = \begin{pmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} & 0 & 0 & -\frac{1}{\rho} & 0 & 0 \\ 0 & -s_{xy}B & \rho c^2 - s_{yy}B & v & 0 & 0 & 0 & 0 & B \\ 0 & -2s_{xy} & \frac{2}{3}\mu & 0 & v & 0 & 0 & 0 & 0 \\ 0 & -\rho c_{2y}^2 & 0 & 0 & 0 & v & 0 & 0 & 0 \\ 0 & 2s_{xy} & -\frac{4}{3}\mu & 0 & 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ k_\rho & 0 & 0 & k_p & 0 & 0 & 0 & 0 & v \end{pmatrix}.$$

The eigenvalues of the A_x and A_y matrices are as follows:

$$\begin{gathered} u - c_{1x},\; u - c_{2x},\; u - c_{3x},\; u,\; u,\; u,\; u + c_{3x},\; u + c_{2x},\; u + c_{1x}, \\ v - c_{1y},\; v - c_{2y},\; v - c_{3y},\; v,\; v,\; v,\; v + c_{3y},\; v + c_{2y},\; v + c_{1y}, \end{gathered} \tag{7}$$

where

$$c_{1x}^2 = \frac{1}{2}\left[c^2 + k_p B + \frac{1}{\rho}\left(\frac{4}{3}\mu - s_{xx}B\right) + A_x\right], \quad c_{2x}^2 = \frac{\mu}{\rho}\left(s_{yy} - s_{xx}\right),$$

$$c_{3x}^2 = \frac{1}{2}\left[c^2 + k_p B + \frac{1}{\rho}\left(\frac{4}{3}\mu - s_{xx}B\right) - A_x\right],$$

$$A_x^2 = \left(k_p B + c^2\right)^2 + 4k_\rho B + \frac{2}{\rho}\left[c^2\left(\frac{4}{3}\mu - s_{xx}B\right) - k_\rho\left(\frac{4}{3}\mu + s_{xx}B\right)\right] + \left[\frac{1}{\rho}\left(\frac{4}{3}\mu - s_{xx}B\right)\right]^2,$$

$$c_{1y}^2 = \frac{1}{2}\left[c^2 + k_p B + \frac{1}{\rho}\left(\frac{4}{3}\mu - s_{yy}B\right) + A_y\right], \quad c_{2y}^2 = \frac{\mu}{\rho}\left(s_{xx} - s_{yy}\right),$$

$$c_{3y}^2 = \frac{1}{2}\left[c^2 + k_p B + \frac{1}{\rho}\left(\frac{4}{3}\mu - s_{yy}B\right) - A_y\right],$$

$$A_y^2 = \left(k_p B + c^2\right)^2 + 4k_\rho B + \frac{2}{\rho}\left[c^2\left(\frac{4}{3}\mu - s_{yy}B\right) - k_\rho\left(\frac{4}{3}\mu + s_{yy}B\right)\right] + \left[\frac{1}{\rho}\left(\frac{4}{3}\mu - s_{yy}B\right)\right]^2.$$

Thus, in the model of a thermally conductive solid, in addition to the elastic (c_1) and shear (c_2) soon-sound networks, there is also a speed of movement of the heat wave (c_3). Eigenvectors corresponding to eigenvalues (7) are linearly independent, so the system of equations in question is also of the hyperbolic type.

3. Godunov hybrid method

To integrate equations (2), let us split the original system into two subsystems. To the first, we attribute the equations written in divergent form, to the second – all the rest. The first subsystem has the form

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}_x}{\partial x}+\frac{\partial \mathbf{F}_y}{\partial y}=0, \tag{8}$$

where

$$\mathbf{U}=\left(\rho,\rho u,\rho v,\rho e\right)^{\mathrm{T}},\quad \mathbf{F}_x=\left[\rho u, p-s_{xx}+\rho u^2,\rho uv-s_{xy},u\left(\rho e+p\right)-us_{xx}-vs_{xy}\right]^{\mathrm{T}},$$
$$\mathbf{F}_y=\left[\rho v,\rho uv-s_{xy},p-s_{yy}+\rho v^2,v\left(\rho e+p\right)-us_{xy}-vs_{yy}\right]^{\mathrm{T}}.$$

The corresponding finite-volume expressions approximating (8) and linking the desired parameters on the new time layer with the corresponding values on the previous t are:

$$\mathbf{U}^{i,j}=\mathbf{U}_{i,j}+\left(\frac{\mathbf{\Phi}_{i-1/2,j}-\mathbf{\Phi}_{i+1/2,j}}{\Delta x_{i,j}}+\frac{\mathbf{\Psi}_{i,j-1/2}-\mathbf{\Psi}_{i,j+1/2}}{\Delta y_{i,j}}\right)\Delta t, \tag{9}$$

where

$$\mathbf{\Phi}=\left[RU,P-S_{xx}+RU^2,RUV-S_{xy},U\left(RE+P\right)-US_{xx}-VS_{xy}\right]^{\mathrm{T}},$$
$$\mathbf{\Psi}=\left[RV,RUV-S_{xy},P-S_{yy}+RV^2,V\left(RE+P\right)-US_{xy}-VS_{yy}\right]^{\mathrm{T}}.$$

"Large" values [4], i.e. R is the medium density; U, V are components of the velocity vector; P – pressure; S_{xx},S_{xy},S_{yy} – components of the voltage tensor deviator related to the faces of adjacent cells, determining from the solution of the corresponding Riemann problems using a linearized Riemannian solver (LRS) [8], the formula for which in the direction of the Ox axis has the form

$$\mathbf{U}_{(L+R)/2}=\mathbf{U}-\frac{1}{2}\sum_k a_k\,\mathrm{sign}\left(\lambda_k\right)\mathbf{X}_k, \tag{10}$$

where $\mathbf{X}_k$ are the right eigenvectors of the matrix A_x are the columns of the first matrix (5),

$$\mathbf{U}_{(L+R)/2}=(\rho,u,v,p,s_{xx},s_{xy},s_{yy})^{\mathrm{T}}_{(L+R)/2},\quad \mathbf{U}=\frac{1}{2}\left(\mathbf{U}_L+\mathbf{U}_R\right),$$
$$\mathbf{U}_L=(\rho,u,v,p,s_{xx},s_{xy},s_{yy})^{\mathrm{T}}_{i-1/2,\,j},\quad \mathbf{U}_R=(\rho,u,v,p,s_{xx},s_{xy},s_{yy})^{\mathrm{T}}_{i+1/2,\,j}.$$

The corresponding LRS formula in the direction of the axis Oy has the form

$$\mathbf{V}_{(D+T)/2}=\mathbf{V}-\frac{1}{2}\sum_k b_k\,\mathrm{sign}\left(\lambda_k\right)\mathbf{Y}_k, \tag{11}$$

where $\mathbf{Y}_k$ are the right eigenvectors of the matrix A_y are the columns of the second matrix (5),

$$\mathbf{V}_{(D+T)/2}=(\rho,u,v,p,s_{xx},s_{xy},s_{yy})^{\mathrm{T}}_{(D+T)/2},\quad \mathbf{V}=\frac{1}{2}\left(\mathbf{V}_D+\mathbf{V}_T\right),$$
$$\mathbf{V}_D=(\rho,u,v,p,s_{xx},s_{xy},s_{yy})^{\mathrm{T}}_{i,\,j-1/2},\quad \mathbf{V}_T=(\rho,u,v,p,s_{xx},s_{xy},s_{yy})^{\mathrm{T}}_{i,\,j+1/2}.$$

The values of the constants a_k and b_k included in (10) and (11) are determined from the systems of linear equations

$$\sum_k a_k \mathbf{X}_k = \Delta\mathbf{U}, \quad \sum_k b_k \mathbf{Y}_k = \Delta\mathbf{V},$$

where $\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L$, $\Delta\mathbf{V} = \mathbf{V}_T - \mathbf{V}_D$. The "large" values that are included in (9) were calculated from the ratios:

$$\left(R, U, V, P, S_{xx}, S_{xy}, S_{yy}\right)^{\mathrm{T}}_{i+1/2,\, j} = \\ = \begin{cases} (\rho, u, v, p, s_{xx}, s_{xy}, s_{yy})^{\mathrm{T}}_{i,\, j}, & \text{если} \quad (u - c_{1x})_{i+1/2, j} > 0, \\ (\rho, u, v, p, s_{xx}, s_{xy}, s_{yy})^{\mathrm{T}}_{i+1,\, j}, & \text{если} \quad (u + c_{1x})_{i+1/2, j} < 0, \\ (\rho, u, v, p, s_{xx}, s_{xy}, s_{yy})^{\mathrm{T}}_{i+1/2,\, j}, & \text{если} \quad (u - c_{1x})_{i+1/2, j} \le 0, \ (u + c_{1x})_{i+1/2, j} \ge 0; \end{cases}$$

$$\left(R, U, V, P, S_{xx}, S_{xy}, S_{yy}\right)^{\mathrm{T}}_{i,\, j+1/2} = \\ = \begin{cases} (\rho, u, v, p, s_{xx}, s_{xy}, s_{yy})^{\mathrm{T}}_{i,\, j}, & \text{если} \quad (u - c_{1y})_{i, j+1/2} > 0, \\ (\rho, u, v, p, s_{xx}, s_{xy}, s_{yy})^{\mathrm{T}}_{i,\, j+1}, & \text{если} \quad (u + c_{1y})_{i, j+1/2} < 0, \\ (\rho, u, v, p, s_{xx}, s_{xy}, s_{yy})^{\mathrm{T}}_{i,\, j+1/2}, & \text{если} \quad (u - c_{1y})_{i, j+1/2} \le 0, \ (u + c_{1y})_{i, j+1/2} \ge 0. \end{cases}$$

To calculate the components of the voltage tensor deviator, write in finite difference form the equations of the second subsystem as

$$\frac{\left(s_{xx}\right)^{i,j} - \left(s_{xx}\right)_{i,j}}{\Delta t} + u_{i,j}\frac{\left(S_{xx}\right)_{i+1/2,j} - \left(S_{xx}\right)_{i-1/2,j}}{\Delta x_{i,j}} + v_{i,j}\frac{\left(S_{xx}\right)_{i,j+1/2} - \left(S_{xx}\right)_{i,j-1/2}}{\Delta y_{i,j}} - \\ -2\left(s_{xy}\right)_{i,j}\left(\frac{U_{i,j+1/2} - U_{i,j-1/2}}{\Delta y_{i,j}} - \frac{V_{i+1/2,j} - V_{i-1/2,j}}{\Delta x_{i,j}}\right) - \frac{2}{3}\mu\left(2\frac{U_{i+1/2,j} - U_{i-1/2,j}}{\Delta x_{i,j}} - \frac{V_{i,j+1/2} - V_{i,j-1/2}}{\Delta y_{i,j}}\right) = -\lambda\left(s_{xx}\right)_{i,j},$$

$$\frac{\left(s_{xy}\right)^{i,j} - \left(s_{xy}\right)_{i,j}}{\Delta t} + u_{i,j}\frac{\left(S_{xy}\right)_{i+1/2,j} - \left(S_{xy}\right)_{i-1/2,j}}{\Delta x_{i,j}} + v_{i,j}\frac{\left(S_{xy}\right)_{i,j+1/2} - \left(S_{xy}\right)_{i,j-1/2}}{\Delta y_{i,j}} - \\ -\left(\rho c_{2x}^2\right)_{i,j}\frac{V_{i+1/2,j} - V_{i-1/2,j}}{\Delta x_{i,j}} - \left(\rho c_{2y}^2\right)_{i,j}\frac{U_{i,j+1/2} - U_{i,j-1/2}}{\Delta y_{i,j}} = -\lambda\left(s_{xy}\right)_{i,j}, \tag{12}$$

$$\frac{\left(s_{yy}\right)^{i,j} - \left(s_{yy}\right)_{i,j}}{\Delta t} + u_{i,j}\frac{\left(S_{yy}\right)_{i+1/2,j} - \left(S_{yy}\right)_{i-1/2,j}}{\Delta x_{i,j}} + v_{i,j}\frac{\left(S_{yy}\right)_{i,j+1/2} - \left(S_{yy}\right)_{i,j-1/2}}{\Delta y_{i,j}} - \\ -2\left(s_{xy}\right)_{i,j}\left(\frac{V_{i+1/2,j} - V_{i-1/2,j}}{\Delta x_{i,j}} - \frac{U_{i,j+1/2} - U_{i,j-1/2}}{\Delta y_{i,j}}\right) - \frac{2}{3}\mu\left(2\frac{V_{i,j+1/2} - V_{i,j-1/2}}{\Delta y_{i,j}} - \frac{U_{i+1/2,j} - U_{i-1/2,j}}{\Delta x_{i,j}}\right) = -\lambda\left(s_{yy}\right)_{i,j},$$

from which are calculated $\left(s_{xx}\right)^{i,j}$, $\left(s_{xy}\right)^{i,j}$, $\left(s_{yy}\right)^{i,j}$, which completes the computational cycle.

4. Numerical simulation results

When testing the GHM described above, a two-way equation of state was used

$$\varepsilon = \frac{p - c_0^2\left(\rho - \rho_0\right)}{\rho\left(\gamma - 1\right)}, \tag{13}$$

previously used by the author in the study on the hydrodynamic approximation of the "oblique" collision of metal plates [14], where for aluminum the constants in equation (13) were relied on as follows: $c_0 =$ 5500 m/s; $\rho_0 =$ 2710 kg/m^3; $\gamma =$ 3,099 [15], which were selected from the condition of approximation of experimental data available in the literature. The yield strength σ_s for aluminum was assumed to be 0,29 GPa and the shear modulus $\mu = 27{,}6$ GPa. For the solid state equation used, the expression for the hydro-dynamic speed of sound is where $c = \sqrt{\dfrac{\gamma\left(p + p_*\right)}{\rho}}$, where $p_* = \dfrac{\rho_0 c_0^2}{\gamma}$.

As the first task, the interaction of an aluminum sample with a rigid barrier is considered, located at the beginning of the coordinate system (x = 0) in the one-dimensional setting. On the left border, the boundary condition for the hard wall gestured $u\big|_{x=0} = 0$, and the right border, at x = 0,1 m, was supposed to be free, through which the striker material can freely flow or flow out (the length of the striker exceeds the time measure of the calculated area). The initial parameters at t = 0 in the task are as follows: density $\rho = \rho_0$, speed $u = u_0$, pressure p_0 = 0, voltage deviation $(s_{ij})_0$ = 0. For this purpose, the wall reflected wave modes described in [3] are known. So, in the speed range from –991 to –34 m/s, a two-wave aging mode with an elastic precursor is implemented, outside this velocity interval there are one-wave reflection modes.

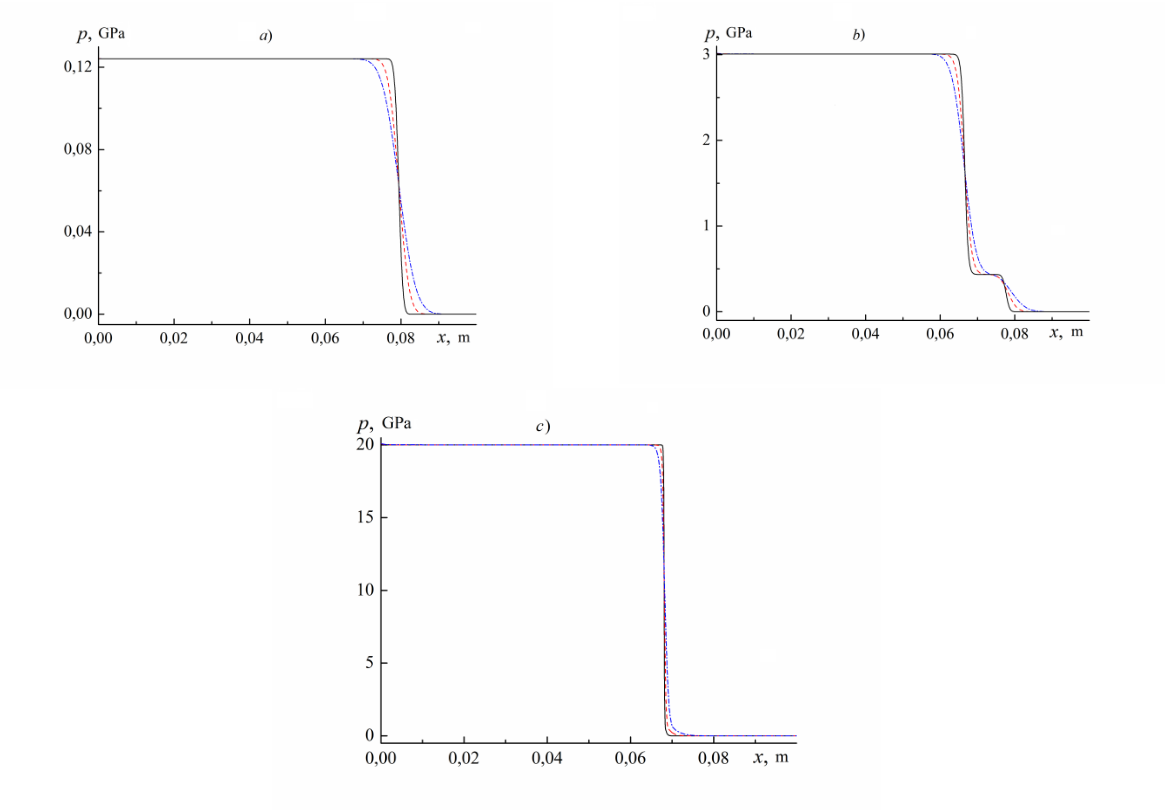

Fig. 1. Pressure distributions by time point t = 12 μs at interaction of aluminum striker with a rigid barrier with speeds:u_0 = –10 (a); –200 (b); –1100 m/s (c). Solid, dashed and dashed curves are obtained on grids of 2000, 1000 and 500 cells

Figure 1 shows the pressure data $p(x)$ obtained at the time t = 12 μs for the characteristic impact speeds from the above sub-areas: u_0 = –10; –200; –1100 m/s. The calculations used a uniform grid consisting of 500, 1000 and 2000 cells. Calculations were carried out at a constant time step $\Delta t = 6\times10^{-9}$ s. As can be seen from the calculations presented, the calculated wave modes correspond to those described in [3].

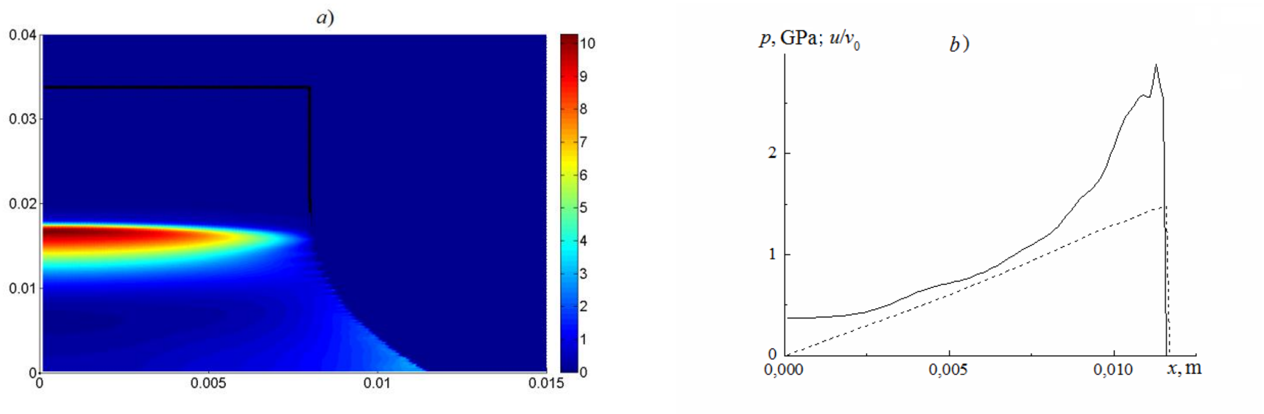

Fig. 2. Hydrodynamic pressure distribution (GPa) at time t = 0.32 μs *a*); dependencies of pressure (continuous curve) and spreading rate (dashed) at barrier *b*) at impact of aluminum striker against rigid barrier

To illustrate the two-dimensional calculation, the problem of interaction of an aluminum striker with a hard barrier with a size of $0{,}8\times4$ cm is considered. Calculations were carried out on a rectangular grid of 150×400 cells with a constant time step $\Delta t = 7\times10^{-9}$ s. The initial and boundary conditions are the same as in the previous task. To localize the contact boundary, the marker method was used [16].

Figure 2, a–b shows the distributions of hydrodynamic pressure, as well as the dependencies of $p(x)$ and $u/v_0(x)$ spreading speed at the barrier at time t = 0,32 μs at impact speed v_0 = 1000 m/s.

5. Conclusion

To integrate the hyperbolic nondivergent system of equations, describing in the framework of the classical Prandtl-Reis model, the elastoplastic deformation of the medium, a Godunov hybrid method is proposed. When calculating stream variables on the faces of adjacent cells, a linearized Riemannian solver is used, the algorithm of which uses the right eigenvectors of the equations of the model. The equations written in divergent form were approximated using finite-volume formulas, and for the others, that is, not reduced to a divergent form equations, finite-difference relations were used. It is confirmed that, if, instead of the Fourier law we take the Maxwell-Cattaneo law, then the GHM can be used to calculate the deformation of a thermally conductive elastoplastic medium, since the equations of the medium model in this case are also of the hyperbolic type. To illustrate the performance of the GHM described in the work, a number of one- and two-dimensional tasks are calculated.

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