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# PLANE LONGITUDINAL WAVES IN A POROUS FLUID-SATURATED MEDIUM WITH A NONLINEAR RELATIONSHIP BETWEEN DEFORMATIONS AND DISPLACEMENTS OF THE LIQUID PHASE

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A mathematical model is presented that describes the propagation of a plane longitudinal wave in a porous fluid-saturated medium, taking into account the geometric nonlinearity of the liquid component of the medium. The nonlinear relationship between deformations and displacements refines the classical Biot theory, within the framework of which a porous fluid-saturated medium is considered. Evolutionary equations for the displacements of the skeleton of the medium and fluid in the pores are obtained. It is shown that if the liquid is confined in the pores, then the wave propagation is described by an equation that generalizes the well-known Burgers equation and has a solution in the form of a stationary shock wave resulting from mutual compensation of the effects of nonlinearity and dissipation. The dependence of the width of the shock wave front on the viscosity of the fluid saturating the pores and the shock wave amplitude is determined. As the viscosity coefficient increases, the wave profile becomes steeper, that is, the wave front width decreases. With an increase in the wave amplitude, the front width can either increase or decrease, depending on the other parameters of the original system. The limiting cases of the obtained generalized Burgers equation are analyzed with respect to the fluid viscosity parameter. If the liquid flows freely in the pores, then the system of evolutionary equations is reduced to a single equation of a simple wave, that is, the propagation of a plane longitudinal wave in a porous medium is described by the well-known equation of nonlinear wave dynamics - the Riemann equation. The equation describes nonlinear waves, which are characterized by a steepening of the leading edge with subsequent overturning resulting from the growth of nonlinear effects in the absence of compensating factors such as dispersion and dissipation.

***Key words***: porous medium (Biot medium), geometric nonlinearity, generalized Burgers equation, stationary shock wave, Riemann wave

Mathematical models of deformable porous media are widely used in the study of processes in geophysics, mechanics of natural and artificial composite materials. In calculations, both classical models dating back to the works by M.A. Biot [1, 2], Ya.I. Frenkel [3], F. Gassman [4], L.Ya. Kosachevsky [5], and their subsequent modifications [6–36] are used.

The authors of the most above mentioned papers restrict their studies to the linear theory of poroelasticity, however, as shown experimentally in [37], nonlinear effects may be significant in porous materials, which is also of particular interest for research. A non-linear generalization of the Biot's model was made in [38].

The equations describing the motion of a porous fluid-saturated medium in a one-dimensional case with regard to the geometric nonlinearity of the solid and fluid phase will be as follows:

$$\begin{cases} \rho_{11}\dfrac{\partial^2 U}{\partial t^2}+\rho_{12}\dfrac{\partial^2 V}{\partial t^2}+b\left(\dfrac{\partial U}{\partial t}-\dfrac{\partial V}{\partial t}\right)=\dfrac{\partial \sigma_{11}}{\partial x} \\ \rho_{12}\dfrac{\partial^2 U}{\partial t^2}+\rho_{22}\dfrac{\partial^2 V}{\partial t^2}+b\left(\dfrac{\partial V}{\partial t}-\dfrac{\partial U}{\partial t}\right)=\dfrac{\partial s}{\partial x} \end{cases}, \qquad (1)$$

where $U=U(x,t)$, $V=V(x,t)$ – are the displacements of the medium skeleton and liquid in pores along the coordinate $x$, $\rho_{11}$, $\rho_{22}$ are the effective densities of the skeleton and fluid in the said pores, $\rho_{12}$ is the mass coupling coefficient between the fluid and the solid phase,

$$\sigma_{11}=Ae_U+2Ne_{11}+Qe_V\,,\;\; s=Qe_U+Re_V\,,$$

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$$e_U = \frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2, \; e_V = \frac{\partial V}{\partial x} + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2, \; e_{11} = \frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2,$$

$A$, $N$, $Q$, $R$ are elastic constants, $b = \left(\eta / K_{pr}\right)\Phi^2$, $\eta$ is dynamic viscosity of the fluid, $\Phi$ is the porosity factor, $K_{pr}$ is the permeability coefficient.

We seek the solution of equations (1) in the form of asymptotic expansions by the small parameter $U = U_{(0)} + \varepsilon U_{(1)} + \ldots$, $V = V_{(0)} + \varepsilon V_{(1)} + \ldots$, where $\varepsilon << 1$. We shall introduce thereat new variables $\xi = x - ct$, $\tau = \varepsilon t$. This choice of variables is explained by the fact that the perturbation propagating with the velocity $c$ along the axis $x$ slowly evolves in the course of time because of the nonlinearity, dispersion, and dissipation. We assume that the nonlinearity in system (1) has the first order of smallness in $\varepsilon$ and the viscosity is a small value.

We restrict herein ourselves to considering the nonlinearity only with respect to the fluid displacement function $V(\xi, \tau)$.

In the zero-order approximation for $\varepsilon$ we obtain the following system of equations:

$$\begin{cases} \left(\rho_{11}c^2 - A - 2N\right)\dfrac{\partial^2 U}{\partial \xi^2} + \left(\rho_{12}c^2 - Q\right)\dfrac{\partial^2 V}{\partial \xi^2} = 0 \\ \left(\rho_{12}c^2 - Q\right)\dfrac{\partial^2 U}{\partial \xi^2} + \left(\rho_{22}c^2 - R\right)\dfrac{\partial^2 V}{\partial \xi^2} = 0 \end{cases}.$$

The condition for the existence of a non-zero solution for this system is given by the following biquadratic equation for determining the velocity

$$m_1 c^4 + m_2 c^2 + m_3 = 0,$$

where $m_1 = \rho_{11}\rho_{22} - \rho_{12}^2$, $m_2 = 2\rho_{12}Q - \rho_{11}R - \rho_{22}\left(A + 2N\right)$, $m_3 = R\left(A + 2N\right) - Q^2$. For the existence of two real and positive roots of the equation, the satisfaction of inequalities for one of the two systems shall be necessary

$$\begin{cases} m_1 > 0 \\ m_2 < 0 \\ m_3 > 0 \end{cases} \text{ or } \begin{cases} m_1 < 0 \\ m_2 > 0 \\ m_3 < 0 \end{cases}.$$

The relationships of the adjoint mass density $\rho_{12}$ with true phase densities are given, for example, in [2] look as follows $\rho_{11} = \rho_1 - \rho_{12}$, $\rho_{22} = \rho_2 - \rho_{12}$, $\rho_{12} < 0$, $\rho_1$, $\rho_2$ are the densities of the solid and liquid phases, respectively. As is evident, $m_1 > 0$.

The first approximation for $\varepsilon$ reduces to the following system of evolutionary equations

$$\begin{cases} 2\varepsilon c\left(\rho_{11} + \rho_{12}\right)\dfrac{\partial W}{\partial \tau} + 2\varepsilon c\left(\rho_{12} + \rho_{22}\right)\dfrac{\partial G}{\partial \tau} + \left(Q + R\right)G\dfrac{\partial G}{\partial \xi} = 0 \\ cb\left(\rho_{12} + \rho_{11}\right)W - cb\left(\rho_{12} + \rho_{11}\right)G - 2\varepsilon c\left(\rho_{11}\rho_{22} - \rho_{12}^2\right)\dfrac{\partial G}{\partial \tau} + \left(\rho_{12}Q - \rho_{11}R\right)G\dfrac{\partial G}{\partial \xi} = 0 \end{cases}. \quad (2)$$

The following designations are herein introduced: $W = \dfrac{\partial U}{\partial \xi}$, $G = \dfrac{\partial V}{\partial \xi}$.

The system of equations (2) can be reduced here to a single equation with respect to the longitudinal deformation function $G$:

$$\begin{aligned} &4\varepsilon^2 c\left(\rho_{11}\rho_{22}-\rho_{12}^2\right)\frac{\partial^2 G}{\partial \tau^2}+2\varepsilon bc\left(\rho_{11}+\rho_{22}+2\rho_{12}\right)\frac{\partial G}{\partial \tau}+ \\ &+b\left(Q+R\right)G\frac{\partial G}{\partial \xi}+2\varepsilon\left(\rho_{11}R-\rho_{12}Q\right)\frac{\partial}{\partial \tau}\left(G\frac{\partial G}{\partial \xi}\right)=0 \end{aligned} . \quad (3)$$

It is known that the classical equation to describe waves in a nonlinear nondispersive medium with dissipation is the Burgers' equation [39, 40]. It is seen from the obtained equation (3) that there is no dispersion in the medium under the study. Equation (3) is distinguished from the Burgers' equation by the second time derivative and the availability of one more quadratic nonlinear term (the time derivative against the quadratic nonlinearity in the Burgers' equation). Dissipative terms are present explicitly and implicitly in equation (3). The only dissipative term is explicitly included in the equation. The additional nonlinear term with respect to the Burgers' equation partially manifests itself as dissipative. That is seen from the linear approximation of relatively small perturbations. The availability of nonlinear and dissipative components in the equation (3) enables us to make the assumption on the possible existence of stationary shock waves in the medium.

We consider now stationary wave $G = G(\chi)$, where $\chi = \xi - v\tau$ is a travelling coordinate. Equation (3) will be as follows:

$$v^2\frac{d^2 G}{d\chi^2}-a_1 v\frac{d}{d\chi}\left(G\frac{dG}{d\chi}\right)+2a_2 bG\frac{dG}{d\chi}-a_3 bv\frac{dG}{d\chi}=0$$

or

$$\frac{d}{d\chi}\left(v^2\frac{dG}{d\chi}-a_1 vG\frac{dG}{d\chi}+a_2 bG^2-a_3 bvG\right)=0, \quad (4)$$

where $a_1=\dfrac{\rho_{11}R-\rho_{12}Q}{2c\varepsilon\left(\rho_{11}\rho_{22}-\rho_{12}^2\right)}$, $a_2=\dfrac{Q+R}{8c\varepsilon^2\left(\rho_{11}\rho_{22}-\rho_{12}^2\right)}$, $a_3=\dfrac{\rho_{11}+\rho_{22}+2\rho_{12}}{2\varepsilon\left(\rho_{11}\rho_{22}-\rho_{12}^2\right)}$.

The latter equation is similar with accuracy up to coefficients to the equation shown in [40]. The difference is herein in the last term of the equation (4).

Equation (4) has a solution in the form of a stationary shock wave. By integrating the equation once on the travelling coordinate with allowance for the boundary conditions

$$G(\chi)=\begin{cases} G_1, \ \chi\to-\infty \\ G_2, \ \chi\to+\infty \end{cases},$$

we obtain the following expression for the nonlinear wave velocity

$$v=\frac{a_2}{a_3}\left(G_1+G_2\right). \quad (5)$$

The equation obtained by the single integration shall be further integrated once again with allowance for shock wave velocity (5) by the method of separation of variables. When the integration is repeated, the integration constant is taken as zero. So, we obtain the solution in an implicit form:

$$\chi = \frac{(G_1+G_2)}{a_3^2 b(G_2-G_1)}\Big[\big(-a_1a_3G_1 + a_2(G_1+G_2)\big)\ln(G-G_1) + \big(a_1a_3G_2 - a_2(G_1+G_2)\big)\ln(G_2-G)\Big]. \tag{6}$$

The derivative shall be as follows

$$\frac{dG}{d\chi} = \frac{a_3^2 b(G_2-G)(G-G_1)}{(G_1+G_2)\big(a_2(G_1+G_2)-a_1a_3G\big)},$$

where $G$ is determined from equation (6).

The profile of the $G(\chi)$ solution and the $G'(\chi)$ derivative at different values of the viscosity coefficient $b$ are shown in Fig. 1 and Fig. 2. It can be seen from the Figures that the profile for the solution to equation (6) has the form of an asymmetric kink with respect to the inflection point. When the viscosity coefficient is decreased, the wave profile becomes flatter, i.e. the wave front width increases.

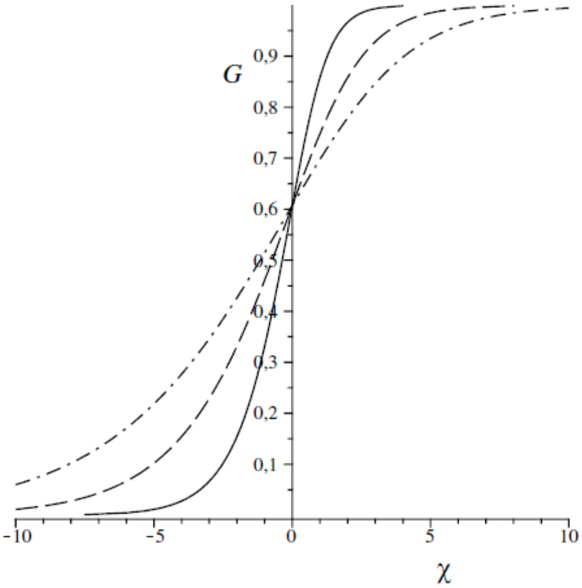


Fig. 1. The profile of stationary shock wave $G(\chi)$ at various viscosity coefficient values shall be $b_1$ (solid line), $b_2$ (dash line), $b_3$ (dash-dotted line), $b_3 < b_2 < b_1$

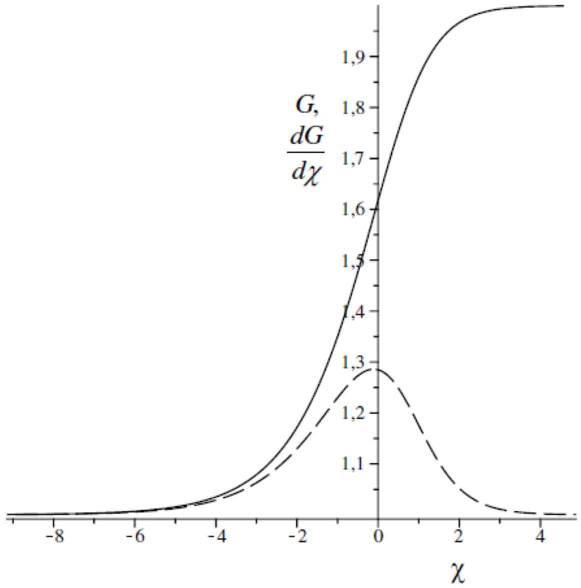


Fig. 2. Dependences of $G(\chi)$ (solid line) and $G'(\chi)$ (dash line, the graph is shifted along the ordinate axis upwards by $G_1$)

Parameters of a shock wave resulting from the mutual compensation of nonlinear and dissipative effects are linked by the ratio

$$\left(a_1 - \frac{v}{A}\right)\frac{v}{a_2 b\Delta} = \text{const}, \tag{7}$$

where $A = G_2 - G_1$ is the shock wave amplitude, $\Delta$ is the characteristic width of the shock wave front and the shock wave velocity $v$ is determined by expression (5).

It can be seen from relationship (7) that the dependence of the wave front width on the fluid viscosity is inversely proportional. The shock wave front width shall depend on the amplitude in inverse proportion with $a_1 < v/A$ and in direct proportion with $a_1 > v/A$. The relationship of the shock wave parameters is different from that of the stationary shock wave in the Burgers' equation. In the classical Burgers' equation the front width is in the direct-proportion dependence on the viscosity.

As $b$ tends to zero, i.e. when the viscosity is small to negligible and fluid flows freely through pores, equation (3), upon the further integration on $\tau$ is transformed to equation

$$\frac{\partial G}{\partial \tau} + \frac{\rho_{11}R - \rho_{12}Q}{2c\varepsilon\left(\rho_{11}\rho_{22} - \rho_{12}^2\right)} G \frac{\partial G}{\partial \xi} = 0 . \quad (8)$$

Equation (8) is a simple wave equation or the Riemann equation [39–42] and it may be solved by the method of characteristics as a first-order partial differential equation.

As the disturbance described by equation (8) propagates, the wave profile is distorted. The wave leading edge (facing towards its motion direction) becomes steeper as the wave travel distance increases over time and the rear one becomes thereat flatter. Different wave profile sections run at different speeds. The wave breaking will occur when characteristics cross for the first time $t^* = \sqrt{\frac{e}{2a^2}}$ , where $a = \frac{\rho_{11}R - \rho_{12}Q}{2c\varepsilon\left(\rho_{11}\rho_{22} - \rho_{12}^2\right)}$, $e$ Euler number (exponent).

Note that in another limiting case, when $b$ is an infinite large value, we also obtain the Riemann equation.

In nature, processes and technology porous liquid-saturated materials with fluid-filled and chaotically located cavities are frequent. Under certain conditions, such cavities induced by an elastic wave oscillate and affect significantly the patterns of the wave propagation. As shown in [43–45], along with the geometric (nonlinear relationship between deformations and displacements) and physical (nonlinear relationship between stresses and deformations) nonlinearities, it is essential to take into account the cavity nonlinearity.

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